The Leaky Bucket

Moral: Just because you have the equations of motion and the initial conditions does not mean you know how to evolve the system for all time. This situation occurs in even simple physical systems.

Say you have a bucket with a hole at the bottom. Initially, the bucket is full of water, and as time goes on, the water leaks out the bottom until it's empty. We'll derive a differential equation for the height of the water in the bucket as a function of time and see that it does not have a unique solution for the given initial conditions. At this point, one may speculate that this is because the differential equation does not satisfy Lipschitz continuity. However, this isn't true, and given a slight modification to the problem, we'll see that even if Lipschitz continuity is not satisfied, a unique solution can still exist.

Let the bucket have cross sectional area A, and the hole have cross sectional area a. At time t, the height of the water in the bucket is $h(t)$.

(a) Using conservation of volume, what is the equation that relates the rate of change of the height of water in the bucket $\dot{h}(t)$ to the velocity of the water coming out of the hole $v(t)$? Assume the water is incompressible and there are no cohesion effects.

(b) Using conservation of energy, what is the relation between the height of the water in the bucket $h(t)$ and the velocity of the water coming out of the hole $v(t)$? You can assume (1) the change in the kinetic energy of the water in the bucket is negligible, and (2) the change in kinetic energy of the water coming out of the bucket is negligible. (Under what circumstances are these reasonable assumptions?)

(c) If you combine the equations in parts (a) and (b), you should find:

$$\frac{dh(t)}{dt} = -C\sqrt{h(t)} \tag{1}$$

What is the expression of C, in terms of the constants in the problem?

(d) Say you're walking by at $t = 0$, you see the bucket and notice it's empty. Can you deduce how much water was in the bucket at some earlier time? Of course not. How can this be shown by solving the differential equation for $h(t)$ in Eq. (1)?

(e) In part (d), you should have shown that there exists a solution to Eq. (1), but it is not unique for all time. What condition of the Picard–Lindelöf theorem does this differential equation fail to satisfy?

(f) Now consider that there is a hose, filling up the bucket. Show that the differential equation is modified to have the following form:

$$\frac{dh(t)}{dt} = -C\sqrt{h(t)} + r \tag{2}$$

To derive this, one can assume the water from the hose only contributes to the height of the water in the bucket, i.e., it does not have to be taken into account when assessing energy conservation. What is the interpretation of r? Now, despite the right-hand side failing to be Lipschitz continuous at $h = 0$, the differential equation indeed can have a unique solution! Can you argue that this is indeed the case?

This is my version of Sean Cornelius's version of Steven Strogatz's version of a problem from Hubbard and West.