This document uses the setup for the calculation done in the Appendix in our paper 1711.07491, cleaning it up a bit, and keeping the effect of recoil through the end, in hopes of deriving the neutrino oscillation formula directly from QFT.

Ingredients

The following enumerates the ingredients that go into the calculation:

(1) **The source.** The source particle that produces the neutrino needs to be localized enough in space so the baseline L of the experiment can be well approximated, but not too localized where the momentum spread of the source becomes significant when calculating quantum amplitudes. This is a semi-classical limit. The following details of how such a state can be constructed. Let the initial state of the source particle have a Gaussian wave function, centered at the origin, with velocity $\mathbf{v}$:

$$|\psi(t=0)\rangle = \int d^d\mathbf{x}\ \psi(t=0,\mathbf{x})\,|\mathbf{x}\rangle_{\rm NR} \tag{1}$$

$$= \int d^d\mathbf{x}\left(\frac{1}{\sqrt{2\pi}\Delta x}\right)^{d/2} e^{-\frac{\mathbf{x}^2}{4\Delta x^2}+i\gamma M\mathbf{v}\cdot\mathbf{x}}\,|\mathbf{x}\rangle_{\rm NR} \tag{2}$$

This is what we'll use as the initial state of the source's wavefunction. Here, M is the mass of the source particle, d is the number of spatial dimensions, $\gamma = 1/\sqrt{1-\mathbf{v}^2}$, and NR indicates a non-relativistic state with momentum state normalizations:

$$|\mathbf{p}\rangle_{\rm NR} = \int d^d\mathbf{x}\ e^{i\mathbf{p}\cdot\mathbf{x}}\,|\mathbf{x}\rangle_{\rm NR}\,, \qquad |\mathbf{x}\rangle_{\rm NR} = \int \frac{d^d\mathbf{p}}{(2\pi)^d}\ e^{-i\mathbf{p}\cdot\mathbf{x}}\,|\mathbf{p}\rangle_{\rm NR} \tag{3}$$

The conjugate representation of Eq. (2) is:

$$|\psi(t=0)\rangle = \int \frac{d^d\mathbf{p}}{(2\pi)^d}\left(2\sqrt{2\pi}\Delta x\right)^{d/2} e^{-\Delta x^2(\mathbf{p}-\gamma M\mathbf{v})^2}\,|\mathbf{p}\rangle_{\rm NR} \tag{4}$$

This wavefunction will need to be evolved in time in the calculation, but we'll choose to make a heavy-particle approximation, which turns off the free diffusion of the source's wavefunction with time. Under free time evolution, the relativistic Hamiltonian is $\hat{\mathbf{H}} = \sqrt{M^2+\hat{\mathbf{P}}^2}$:

$$|\psi(t)\rangle = e^{-it\sqrt{M^2+\hat{\mathbf{P}}^2}}\,|\psi(t=0)\rangle \tag{5}$$

$$= \left(2\sqrt{2\pi}\Delta x\right)^{d/2}\int \frac{d^d\mathbf{p}}{(2\pi)^d}\ e^{-it\sqrt{M^2+\mathbf{p}^2}-\Delta x^2(\mathbf{p}-\gamma M\mathbf{v})^2}\,|\mathbf{p}\rangle_{\rm NR} \tag{6}$$

$$= \left(2\sqrt{2\pi}\Delta x\right)^{d/2}\int d^d\mathbf{x}\int \frac{d^d\mathbf{p}}{(2\pi)^d}\ e^{-it\sqrt{M^2+\mathbf{p}^2}-\Delta x^2(\mathbf{p}-\gamma M\mathbf{v})^2+i\mathbf{p}\cdot\mathbf{x}}\,|\mathbf{x}\rangle_{\rm NR} \tag{7}$$

where $\hat{\mathbf{P}}\,|\mathbf{p}\rangle_{\rm NR} = \mathbf{p}\,|\mathbf{p}\rangle_{\rm NR}$. So far, this is general for a Gaussian wavefunction. Now we can make experimentally-relevant approximations, using a technique from HQET. Take the argument of the exponent of the integrand, let $\mathbf{p} = \gamma M\mathbf{v}+\mathbf{k}$, and expand in powers of $\mathbf{k}$:

$$-it\sqrt{M^2+\mathbf{p}^2} - \Delta x^2(\mathbf{p}-\gamma M\mathbf{v})^2 + i\mathbf{p}\cdot\mathbf{x} \tag{8}$$

$$= -i\gamma M(t-\mathbf{v}\cdot\mathbf{x}) + i(\mathbf{x}-\mathbf{v}t)\cdot\mathbf{k} - \Delta x^2\mathbf{k}^2 + \mathcal{O}\left(\frac{t}{M\Delta x^2}\right) \tag{9}$$

The neglected higher-order terms are suppressed by terms that scale like $t/((M\Delta x)^n \Delta x)$, where n is some integer greater than zero. This is because the $e^{-\Delta x^2 \mathbf{k}^2}$ term suppresses the large-k region, so $k \sim 1/\Delta x$. Ignoring these terms is the approximation that the spatial uncertainly of the source's wave function is much larger than its Compton wavelength for the duration of the experiment. This removes the free-particle diffusion of the source.

Performing the Gaussian integral over $\mathbf{k}$:

$$|\psi(t)\rangle = \left(\frac{1}{\sqrt{2\pi}\Delta x}\right)^{d/2} \int d^d\mathbf{x}\ e^{-i\gamma M(t-\mathbf{v}\cdot\mathbf{x})-\frac{(\mathbf{x}-\mathbf{v}t)^2}{4\Delta x^2}}\,|\mathbf{x}\rangle_{\rm NR} \tag{10}$$

$$\equiv \int d^d\mathbf{x}\ \psi(t,\mathbf{x})\,|\mathbf{x}\rangle_{\rm NR} \tag{11}$$

We'll use this representation of the wave function at time t in the calculation.

Lastly, the lifetime of the source can be taken into account by including $-\Gamma t$ in the exponent of the source's wavefunction. However, the following calculation will assume $\Gamma = 0$.

(2) **The neutrino.** Neutrinos are fermions, but this calculation will treat them like scalars. The full interaction probability between source and detector does depend on the spin of the propagating particle between them. However, since we're only interested in the neutrino oscillation probability, only the location of the neutrino's pole matters, and its spin indices have no effect on the oscillation behavior. The reason for this is that the location of the pole for a propagating fermionic degree of freedom is the same as that of a scalar's with the same mass. To see this, let $G_S(p)$ be a scalar Green's function with pole at $p^2 = m^2$, then the fermionic Green's function for a particle with the same mass is $G_F(p) = (i\gamma \cdot \partial + m)G_S(p)$, which therefore must also have a pole at $p^2 = m^2$.

The rotating wave approximation is used to represent the neutrino in the Hamiltonian. To do this, instead of the full free quantum field $\phi_a(x)$ (where the index a stands for the eigenstate with mass m_a), one separates it into pieces with positive and negative phases, $\phi_a(x) = \phi_a^{(+)} + \phi_a^{(-)}$, where

$$\phi_a^{(+)}(t,\mathbf{x}) = \int \frac{d^d\mathbf{p}}{(2\pi)^d\sqrt{2E^a_{\mathbf{p}}}}\ a^\dagger_{\mathbf{p},a}\ e^{iE^a_{\mathbf{p}}t - i\mathbf{p}\cdot\mathbf{x}} \tag{12}$$

$$\phi_a^{(-)}(t,\mathbf{x}) = \int \frac{d^d\mathbf{p}}{(2\pi)^d\sqrt{2E^a_{\mathbf{p}}}}\ a_{\mathbf{p},a}\ e^{-iE^a_{\mathbf{p}}t + i\mathbf{p}\cdot\mathbf{x}} \tag{13}$$

where $E^a_{\mathbf{p}} \equiv \sqrt{\mathbf{p}^2 + m_a^2}$, and $a^{(\dagger)}_{\mathbf{p},a}$ is the annihilation (creation) operator for a neutrino state with momentum $\mathbf{p}$ and mass m_a, with the following commutation relations for bosons: $[a_{\mathbf{k},a'}, a^\dagger_{\mathbf{p},a}] = (2\pi)^d\delta^d(\mathbf{k}-\mathbf{p})\delta_{a'a}$. The rotating wave approximation is that $\phi_a^{(+)}$ and $\phi_a^{(-)}$ appear separately in the Hamiltonian, instead of only the linear combination $\phi_a = \phi_a^{(+)} + \phi_a^{(-)}$.

There are multiple neutrino species, each with mass m_a. In a given weak interaction, a particular superposition of neutrino states are produced $\sum_a U^*_{\ell a}\phi_a^{(+)}$, where $U^*_{\ell a}$ are elements a unitary matrix, so $\sum_a U^*_{\ell a}U_{a\ell'} = \delta_{\ell'\ell}$, and the index ℓ refers to the lepton flavor of the production process. Likewise the weak process of the detection absorbs a particular linear combination of neutrino states $\sum_a U_{\ell' a}\phi_a^{(-)}$. Because the neutrinos are internal particles in oscillation experiments, the contributions from different mass species need to be summed at the amplitude level. If any decoherence occurs, this must be the outcome of the calculation, e.g., when the source has a non-negligible life time, as we showed in 1711.07491.

(3) **The production process.** A single source particle with mass M decays to a neutrino with mass m_a and a collection of other particles X, with invariant mass M_X. This process is symbolized as $S \to X+\nu$. It's typical in semileptonic processes to factorize the operator that transitions between the initial S state and the final $X+\nu$ state between the operator that gives rise to neutrino production and the non-perturbative operator $\mathcal{O}_{S\to X}$ that transitions between S and X, where the interaction Hamiltonian for production is:

$$V_{\rm prod}(t) = \int d^d\mathbf{x} \, \sum_a U^*_{\ell a}\phi_a^{(+)}(t,\mathbf{x}) \; \mathcal{O}_{S\to X}(t,\mathbf{x}) + \text{h.c.} \tag{14}$$

Here, $\mathcal{O}_{S\to X}(t,\mathbf{x}) = e^{i\hat{\mathbf{H}}t - i\hat{\mathbf{P}}\cdot\mathbf{x}}\mathcal{O}_{S\to X}(0,0)e^{-i\hat{\mathbf{H}}t+i\hat{\mathbf{P}}\cdot\mathbf{x}}$, and the only non-zero matrix elements of the local operator are

$$\langle X,\mathbf{p}_X|\,\mathcal{O}_{S\to X}(t,\mathbf{x})\,|S,\mathbf{p}_S\rangle \;=\; e^{i(E_X-E_S)t-i(\mathbf{p}_X-\mathbf{p}_S)\cdot\mathbf{x}}\;F(X,S;\mathbf{p}_S,\mathbf{p}_X) \tag{15}$$

where $|X,\mathbf{p}_X\rangle$ and $|S,\mathbf{p}_S\rangle$ are the states with well-defined momentum (the labels X and S represent all the other information about the state, e.g., its spin, invariant mass, angular distributions, etc.), and F is a form factor, i.e., $F(X,S;\mathbf{p}_S,\mathbf{p}_X) \equiv \langle X,\mathbf{p}_X|\,\mathcal{O}_{S\to X}(0,\mathbf{0})\,|S,\mathbf{p}_S\rangle$, which depends on all the details of the $S\to X$ transition. Also, $E_X = \sqrt{M_X^2+\mathbf{p}_X^2}$ and $E_S = \sqrt{M^2+\mathbf{p}_S^2}$.

We'll make the following simplification: the differences in the neutrino masses are small enough that the form factor F is approximately the same for all neutrino species. With this approximation, we'll let

$$g_S \equiv F(X,S;\mathbf{p}_S,\mathbf{p}_X) \tag{16}$$

be a constant for all neutrino mass eigenstates for a single weak transition.

(4) **The detection process.** The neutrino is detected by a heavy two-state system with energy level splitting Δ, located at spatial position $\mathbf{L}$. Here, the neutrino is absorbed, and the detector transitions from the ground state $|G\rangle$ to the excited state $|E\rangle$. The detection Hamiltonian is:

$$V_{\rm det}(t) = g_D \sum_a U_{a\ell'}\phi_a^{(-)}(t,\mathbf{L})\; e^{i\Delta t}\,|E\rangle\,\langle G| + \text{h.c} \tag{17}$$

where g_D is a coupling constant, taken to be the same value for all neutrino species.

There are two additional ingredients that make this detector model more realistic. First, we should set up an entire collection of 2-state systems, all located at spatial position $\mathbf{L}$, each with a slightly different energy splitting Δ_j. The probability density of these energy splittings is $\rho(\Delta_j)$. These states have inner products $\langle G_j|G_k\rangle = \delta_{jk}$ and $\langle E_j|E_k\rangle = \delta_{jk}$. Each of these detectors are distinguishable, so the total transition probability involves an incoherent sum over these these detector subsystems.

Second, the detector also measures the time of the neutrino detection. To account for this, all the detectors will be on only during the time interval $[\tau, \tau + T]$, where T corresponds to the timing resolution of the detector. We will be taking T to be large enough to ensure energy conservation at the detector, i.e., $\Delta_j T \gg 1$ which is a very good approximation of current particle-physics detector technology

(5) **The Hamiltonian**. The full interaction Hamiltonian is

$$\begin{aligned} V(t) &= V_{\text{prod}}(t) + V_{\text{det}}(t) && (18) \\ &= \sum_a \int d^d\mathbf{x} \Big[U^*_{\ell a}\ \phi_a^{(+)}(t, \mathbf{x})\ \mathcal{O}_{S\to X}(t, \mathbf{x}) \Big] \\ &\quad + g_D \sum_j \sum_a U_{a\ell'}\ \phi_a^{(-)}(t, \mathbf{L})\ e^{i\Delta_j t} \left|E_j\right\rangle \left\langle G_j\right| + \text{h.c.} && (19) \end{aligned}$$

(6) **The initial state.** We factorize the Hilbert space between the source, a part of the detector system, and the neutrino. The initial state of the system is:

$$|i\rangle = |\psi(t = 0)\rangle \otimes |G_j\rangle \otimes |0\rangle_\phi \tag{20}$$

Here, the source begins at $t = 0$, the detector subsystem j is in the ground state, and there is no neutrino.

(7) **The final state.** The final state is

$$|f\rangle = |\mathbf{p}_X\rangle \otimes |E_j\rangle \otimes |0\rangle_\phi \tag{21}$$

Here, the X state has momentum $\mathbf{p}_X$, the detector subsystem j is in the excited state, and there is no neutrino.

(8) **The transition amplitude.** Using time-dependent perturbation theory, the lowest-order non-zero amplitude between the initial and final states is

$$\mathcal{A}_{\ell\to\ell'}(\mathbf{p}_X, \Delta_j) = (-i)^2 \int_\tau^{\tau+T} dt_1 \int_0^{t_1} dt_2\ \langle f|\, V(t_1)V(t_2)\, |i\rangle \tag{22}$$

(9) **The sum over final states.** The total probability for the process can be calculated by summing incoherently the different final states of the X state and the detector system:

$$\mathcal{P}_{\ell\to\ell'} = \int \frac{d^d\mathbf{p}_X}{(2\pi)^d} \int d\Delta_j\ \rho(\Delta_j) \left|\mathcal{A}_{\ell\to\ell'}(\mathbf{p}_X, \Delta_j)\right|^2 \tag{23}$$

The rest of this document is the evaluation of Eqs. (22) and (23). There are no more choices left, the rest is math. Note that nowhere above was energy or momentum conservation mentioned, and not even that the neutrino goes from the source to the detector. These are conclusions of the following calculation, not their inputs.

The Amplitude

Starting with the transition amplitude:

$$\mathcal{A}_{\ell\to\ell'}(\mathbf{p}_X,\Delta_j) = (-i)^2\int_{\tau}^{\tau+T}dt_1\int_0^{t_1}dt_2\,\langle f|\,V(t_1)V(t_2)\,|i\rangle \tag{24}$$

$$= -g_D\sum_{a,b}U^*_{\ell a}U_{b\ell'}\int_{\tau}^{\tau+T}dt_1\int_0^{t_1}dt_2\int d^d\mathbf{x} \tag{25}$$
$$\times\, e^{i\Delta_j t_1}\,\langle\mathbf{p}_X|\,\mathcal{O}_{S\to X}(t_2,\mathbf{x})\,|\psi(t=0)\rangle\,\langle 0|\,\phi_b^{(-)}(t_1,\mathbf{L})\phi_a^{(+)}(t_2,\mathbf{x})\,|0\rangle$$

Next we'll evaluate the two matrix elements present in the integrand. For the matrix element for the $S\to X$ transition:

$$\langle\mathbf{p}_X|\,\mathcal{O}_{S\to X}(t_2,\mathbf{x})\,|\psi(t=0)\rangle = \left(2\sqrt{2\pi}\Delta x\right)^{d/2}\int\frac{d^d\mathbf{p}_S}{(2\pi)^d}e^{-\Delta x^2(\mathbf{p}_S-\gamma M\mathbf{v})^2}\,\langle\mathbf{p}_X|\,\mathcal{O}_{S\to X}(t_2,\mathbf{x})\,|\mathbf{p}_S\rangle \tag{26}$$

$$= g_S\left(2\sqrt{2\pi}\Delta x\right)^{d/2}\int\frac{d^d\mathbf{p}_S}{(2\pi)^d}\,e^{i(E_X-E_S)t_2-i(\mathbf{p}_X-\mathbf{p}_S)\cdot\mathbf{x}-\Delta x^2(\mathbf{p}_S-\gamma M\mathbf{v})^2} \tag{27}$$

Where we used Eq. (4), (15), and (16). Letting $\mathbf{p}_S=\gamma M\mathbf{v}+\mathbf{k}$, using the approximation in Eq. (9) to approximate the argument of the exponent, and doing the remaining Gaussian integral over $\mathbf{k}$:

$$\langle\mathbf{p}_X|\,\mathcal{O}_{S\to X}(t_2,\mathbf{x})\,|\psi(t=0)\rangle = g_S\left(\frac{1}{\sqrt{2\pi}\Delta x}\right)^{d/2}e^{iE_Xt_2-i\mathbf{p}_X\cdot\mathbf{x}-i\gamma M(t_2-\mathbf{v}\cdot\mathbf{x})-\frac{(\mathbf{x}-\mathbf{v}t_2)^2}{4\Delta x^2}} \tag{28}$$

$$\tag{29}$$

Second, evaluating the remaining matrix element for the neutrino subspace using Eqs. (12) and (13):

$$\langle 0|\,\phi_b^{(-)}(t_1,\mathbf{L})\phi_a^{(+)}(t_2,\mathbf{x})\,|0\rangle = \int\frac{d^d\mathbf{p}}{(2\pi)^d\sqrt{2E^b_{\mathbf{p}}}}\frac{d^d\mathbf{p}'}{(2\pi)^d\sqrt{2E^a_{\mathbf{p}'}}}e^{-iE^b_{\mathbf{p}}t_1+i\mathbf{p}\cdot\mathbf{L}+iE^a_{\mathbf{p}}t_2-i\mathbf{p}'\cdot\mathbf{x}}\,\langle 0|\,a_{\mathbf{p},b}a^\dagger_{\mathbf{p}',a}\,|0\rangle \tag{30}$$

$$= \delta_{ab}\int\frac{d^d\mathbf{p}}{(2\pi)^d}\frac{1}{2\sqrt{E^b_{\mathbf{p}}E^a_{\mathbf{p}}}}\;e^{-iE^b_{\mathbf{p}}t_1+i\mathbf{p}\cdot\mathbf{L}+iE^a_{\mathbf{p}}t_2-i\mathbf{p}\cdot\mathbf{x}} \tag{31}$$

Using the expressions in Eqs. (28) and (31) in Eq. (25), and doing the sum over b, we have:

$$\mathcal{A}_{\ell\to\ell'}(\mathbf{p}_X,\Delta_j) = -g_Dg_S\left(\frac{1}{\sqrt{2\pi}\Delta x}\right)^{d/2}\sum_a U^*_{\ell a}U_{a\ell'}\int_{\tau}^{\tau+T}dt_1\int_0^{t_1}dt_2\int d^d\mathbf{x}\int\frac{d^d\mathbf{p}}{(2\pi)^d}\frac{1}{2E^a_{\mathbf{p}}}$$
$$\times\left[\,e^{i(\Delta_j-E^a_{\mathbf{p}})t_1+i(E_X+E^a_{\mathbf{p}}-\gamma M)t_2-i(\mathbf{p}+\mathbf{p}_X-\gamma M\mathbf{v})\cdot\mathbf{x}+i\mathbf{p}\cdot\mathbf{L}-\frac{(\mathbf{x}-\mathbf{v}t_2)^2}{4\Delta x^2}}\right] \tag{32}$$

Doing the integral over $\mathbf{x}$:

$$\mathcal{A}_{\ell\to\ell'}(\mathbf{p}_X,\Delta_j) = -g_Dg_S\left(2\sqrt{2\pi}\Delta x\right)^{d/2}\sum_a U^*_{\ell a}U_{a\ell'}\int_{\tau}^{\tau+T}dt_1\int_0^{t_1}dt_2\int\frac{d^d\mathbf{p}}{(2\pi)^d}\frac{1}{2E^a_{\mathbf{p}}} \tag{33}$$
$$\times\left[\,e^{i(\Delta_j-E^a_{\mathbf{p}})t_1+i(E_X+E^a_{\mathbf{p}}-\gamma M)t_2-i\mathbf{v}\cdot(\mathbf{p}+\mathbf{p}_X-\gamma M\mathbf{v})t_2+i\mathbf{p}\cdot\mathbf{L}-\Delta x^2(\mathbf{p}+\mathbf{p}_X-\gamma M\mathbf{v})^2}\right]$$

Let $d = 3$, $\mathbf{L} = L\hat{z}$, and express the $\mathbf{p}$ integral in terms of spherical coordinates, where $\theta = 0$ corresponds to the $\hat{z}$ axis. Let $x = \cos\theta$. Isolating the just angular integrals,

$$\int_{-1}^{1} dx \int_{0}^{2\pi} d\phi \; e^{-ip\sqrt{1-x^2}(v_x\cos\phi + v_y\sin\phi)t_2 - ipv_z x t_2 + ipLx} \tag{34}$$

$$\times \; e^{-2p\Delta x^2\left(\sqrt{1-x^2}((p_X^x - \gamma M v_x)\cos\phi + (p_X^y - \gamma M v_y)\sin\phi) + x(p_X^z - \gamma M v_z)\right)} \tag{35}$$

One can use the following approximation:

$$\int_{-1}^{1} dx \int_{0}^{2\pi} d\phi \; e^{iax \pm cx + b\sqrt{1-x^2}\cos\phi} \quad \overset{c\to\infty}{\sim} \quad \pm\frac{e^{\pm(ia+c)}}{ia+c} \quad \overset{c/a\to 0}{\sim} \quad \pm\frac{e^{\pm(ia+c)}}{ia} \tag{36}$$

which is accurate if Δx is taken to be large (to approximate the integral), but always remaining much smaller than L (to approximate the result of integration), which is the case in experiments. The same approximation applies if $\cos\phi$ is replaced with $\sin\phi$ in the integrand. The $\pm$ sign in the approximation in Eq. (36) depends on the sign of $p_X^z - \gamma M v_z$. One can evaluate the angular integrals keeping track of the sign of $p_X^z - \gamma M v_z$, but I claim that the situation when $p_X^z - \gamma M v_z > 0$ is killed in the upcoming energy integrals – it corresponds with a negative energy configuration, since this is the solution where the neutrino propagates along the $\hat{z}$ axis, but away from the detector, not towards it. This is discussed in a bit more detail in 1711.07491. For now, we'll focus on when $p_X^z - \gamma M v_z < 0$, which corresponds with the neutrino going toward the detector ($\cos\theta = 0$). After using this approximation, the amplitude becomes:

$$\mathcal{A}_{\ell\to\ell'}(\mathbf{p}_X, \Delta_j) \;=\; -g_D g_S \left(2\sqrt{2\pi}\Delta x\right)^{3/2} \sum_a U_{\ell a}^* U_{a\ell'} \int_{\tau}^{\tau+T} dt_1 \int_0^{t_1} dt_2 \int_0^{\infty} \frac{p^2 dp}{(2\pi)^3}\frac{1}{2E_p^a} \tag{37}$$

$$\times \left(\frac{2\pi}{ipL}\right)\left[\, e^{i(\Delta_j - E_p^a)t_1 + i(E_X + E_p^a - \gamma M)t_2 - i\mathbf{v}\cdot(p\hat{z} + \mathbf{p}_X - \gamma M\mathbf{v})t_2 + ipL - \Delta x^2(p\hat{z} + \mathbf{p}_X - \gamma M\mathbf{v})^2}\right]$$

where $E_p^a \equiv \sqrt{p^2 + m_a^2}$. Changing integration variables from p to E_p^a:

$$\mathcal{A}_{\ell\to\ell'}(\mathbf{p}_X, \Delta_j) \;=\; -g_D g_S \left(2\sqrt{2\pi}\Delta x\right)^{3/2} \sum_a U_{\ell a}^* U_{a\ell'} \int_{\tau}^{\tau+T} dt_1 \int_0^{t_1} dt_2 \int_{m_a}^{\infty} \frac{dE_p^a}{(2\pi)^3} \tag{38}$$

$$\times \left(\frac{2\pi}{iL}\right)\left[\, e^{i(\Delta_j - E_p^a)t_1 + i(E_X + E_p^a - \gamma M)t_2 - i\mathbf{v}\cdot(p_a\hat{z} + \mathbf{p}_X - \gamma M\mathbf{v})t_2 + ip_aL - \Delta x^2(p_a\hat{z} + \mathbf{p}_X - \gamma M\mathbf{v})^2}\right]$$

where $p_a \equiv \sqrt{E_p^a - m_a^2}$. Next we'll deform the integrand slightly in the large (positive) t_2 region by adding an $-\epsilon t_2$ to the argument of the exponent, where ϵ is an infinitesimally small positive real number:

$$\mathcal{A}_{\ell\to\ell'}(\mathbf{p}_X, \Delta_j) \;=\; -g_D g_S \left(2\sqrt{2\pi}\Delta x\right)^{3/2} \sum_a U_{\ell a}^* U_{a\ell'} \int_{\tau}^{\tau+T} dt_1 \int_0^{t_1} dt_2 \int_{m_a}^{\infty} \frac{dE_p^a}{(2\pi)^3} \tag{39}$$

$$\times \left(\frac{2\pi}{iL}\right)\left[\, e^{i(\Delta_j - E_p^a)t_1 + i(E_X + E_p^a - \gamma M + i\epsilon)t_2 - i\mathbf{v}\cdot(p_a\hat{z} + \mathbf{p}_X - \gamma M\mathbf{v})t_2 + ip_aL - \Delta x^2(p_a\hat{z} + \mathbf{p}_X - \gamma M\mathbf{v})^2}\right]$$

Doing the t_2 integral:

$$\mathcal{A}_{\ell\to\ell'}(\mathbf{p}_X, \Delta_j) \;=\; -ig_D g_S \left(2\sqrt{2\pi}\Delta x\right)^{3/2} \sum_a U_{\ell a}^* U_{a\ell'} \int_{\tau}^{\tau+T} dt_1 \int_{m_a}^{\infty} \frac{dE_p^a}{(2\pi)^3} \tag{40}$$

$$\times \left(\frac{2\pi}{iL}\right) e^{i\Delta_j t_1 + ip_aL - \Delta x^2(p_a\hat{z} + \mathbf{p}_X - \gamma M\mathbf{v})^2}\left[\frac{e^{-iE_p^a t_1} - e^{i(E_X - M/\gamma - \mathbf{v}\cdot(p_a\hat{z} + \mathbf{p}_X))t_1}}{E_X + E_p^a - M/\gamma - \mathbf{v}\cdot(p_a\hat{z} + \mathbf{p}_X) + i\epsilon}\right]$$

The evaluation of the energy integral reveals the causal structure of the amplitude. Assuming analyticity in E_p^a, the full integrand has no poles, but the two individual terms do, and these poles are the dominant contribution to the individual terms in the integral. Let's look at the first term. If $t_1 - L > 0$, then the first term is basically the same as in 1711.07491, were you have to deform the contour into the lower-half plane, picking up the contribution from the pole. This corresponds to the dominant contribution being when the detector is on within the neutrino's light cone. If instead $t_1 - L < 0$, you deform the contour to the upper-half plane where there is no pole, and the this first term is suppressed. Regarding the second term, it can pick up the contribution from the pole if $L - v_z t_1 < 0$, and it would cancel the pole in the first term when also $t_1 - L > 0$. Taking both terms together, the only non-trivial contribution from the above energy integral is from the pole, and when both the conditions $t_1 - L > 0$ and $L - v_z t_1 > 0$ are satisfied. These are the same results we found in 1711.07491, but when $\mathbf{v} = 0$. The value of E_p^a at the location of the pole, call it E_*^a, satisfies the equation:

$$E_X + E_*^a - M/\gamma - \mathbf{v} \cdot \left(\sqrt{(E_*^a)^2 - m_a^2}\,\hat{z} + \mathbf{p}_X \right) = 0 \tag{41}$$

where, again, $E_X \equiv \sqrt{|\mathbf{p}_X|^2 + M_X^2}$. I won't solve this equation just yet, but instead just note that the root of it is located at E_*^a.

Now performing the energy integral via the residue methods just mentioned:

$$\begin{aligned} \mathcal{A}_{\ell\to\ell'}(\mathbf{p}_X, \Delta_j) &= -ig_D g_S \left(2\sqrt{2\pi}\Delta x\right)^{3/2} \sum_a U_{\ell a}^* U_{a\ell'} \int_\tau^{\tau+T} dt_1 \frac{1}{(2\pi)^3} \\ &\quad \times (-2\pi i) \left(\frac{2\pi}{iL}\right) \, e^{i(\Delta_j - E_*^a)t_1 + iL\sqrt{(E_*^a)^2 - m_a^2} - \Delta x^2 \left(\sqrt{(E_*^a)^2 - m_a^2}\,\hat{z} + \mathbf{p}_X - \gamma M \mathbf{v}\right)^2} \end{aligned} \tag{42}$$

Doing the t_1 integral:

$$\begin{aligned} \mathcal{A}_{\ell\to\ell'}(\mathbf{p}_X, \Delta_j) &= -ig_D g_S \left(2\sqrt{2\pi}\Delta x\right)^{3/2} \sum_a U_{\ell a}^* U_{a\ell'} \frac{e^{i(\tau+T/2)(\Delta_j - E_*^a)}}{(2\pi)^3} \frac{2\sin[(\Delta_j - E_*^a)T/2]}{\Delta_j - E_*^a} \\ &\quad \times (-2\pi i) \left(\frac{2\pi}{iL}\right) \, e^{iL\sqrt{(E_*^a)^2 - m_a^2} - \Delta x^2 \left(\sqrt{(E_*^a)^2 - m_a^2}\,\hat{z} + \mathbf{p}_X - \gamma M \mathbf{v}\right)^2} \end{aligned} \tag{43}$$

And cleaning it up a bit:

$$\begin{aligned} \mathcal{A}_{\ell\to\ell'}(\mathbf{p}_X, \Delta_j) &= i\frac{g_D g_S}{\pi L} \left(2\sqrt{2\pi}\Delta x\right)^{3/2} \sum_a U_{\ell a}^* U_{a\ell'} e^{i(\tau+T/2)(\Delta_j - E_*^a)} \frac{\sin[(\Delta_j - E_*^a)T/2]}{\Delta_j - E_*^a} \\ &\quad \times \, e^{iL\sqrt{(E_*^a)^2 - m_a^2} - \Delta x^2 \left(\sqrt{(E_*^a)^2 - m_a^2}\,\hat{z} + \mathbf{p}_X - \gamma M \mathbf{v}\right)^2} \end{aligned} \tag{44}$$

That's it for the amplitude. No assumptions yet have been made about neutrinos being ultrarelativistic, and nowhere have the words "on-shell" been mentioned.

The Probability

Next we square the amplitude, and do the four integrals over the final states of Δ_j and $\mathbf{p}_X$, as in Eq. (23), in order to get the probability:

$$\mathcal{P}_{\ell\to\ell'} = \int \frac{d^3\mathbf{p}_X}{(2\pi)^3} \int d\Delta_j \; \rho(\Delta_j) \left| \mathcal{A}_{\ell\to\ell'}(\mathbf{p}_X, \Delta_j) \right|^2 \tag{45}$$

Let's look first at the integral over Δ_j. We immediately run into an issue. Squaring the individual sinc function yields the diagonal terms:

$$\left|\frac{\sin[(\Delta_j - E_*^a)T/2]}{\Delta_j - E_*^a}\right|^2 \sim \frac{\pi}{2} T \delta(\Delta_j - E_*^a) \qquad \text{as } \Delta_j T \to \infty \tag{46}$$

but the cross terms would behave like:

$$\frac{\sin[(\Delta_j - E_*^a)T/2]}{(\Delta_j - E_*^a)} \frac{\sin[(\Delta_j - E_*^b)T/2]}{(\Delta_j - E_*^b)} \sim 0 \qquad \text{as } \Delta_j T \to \infty \tag{47}$$

if E_*^a and E_*^b are not equal, i.e., the neutrino species a and b have different energies. The contributions to the amplitude from different neutrino species do not interfere, because the detector was able to resolve their individual energies. Therefore, no oscillations occur. This wasn't an issue in the main part of our paper, because all the neutrinos had the same energy, by construction, so these sinc functions perfectly overlapped. However now when including recoil, the non-overlapping sinc functions become a relevant detail. This result is not surprising, since we'd expect such a decoherence effect if the neutrinos had very different masses. The first thing to explore is the possibility of giving the detector an absorption spectrum, which one can model with a Lorentzian with width Γ_D by replacing $\Delta_j \to \Delta_j + i\Gamma_D/2$, where $\Delta_j \gg \Gamma_D$. The assumptions being made are that $\Gamma_D T \gg 1$, and $|E_a^* - E_b^*|$ is smaller than all other energies scales in the problem, so $\Gamma_D \gg |E_a - E_b|$. When Γ_D is nonzero, then Eq. (47) becomes:

$$\frac{\sin[(\Delta_j + i\Gamma_D/2 - E_*^a)T/2]}{(\Delta_j + i\Gamma_D/2 - E_*^a)} \frac{\sin[(\Delta_j - i\Gamma_D/2 - E_*^b)T/2]}{(\Delta_j - i\Gamma_D/2 - E_*^b)} \sim \frac{e^{\Gamma_D T/2}}{(\Delta_j - \bar{E}_*)^2 + \Gamma_D^2/4} \tag{48}$$

as $\Gamma_D T \to \infty$ and $\Delta_j T \to \infty$. Here, $\bar{E}_*$ is some value of energy that is very close to E_*^a and E_*^b, but at this point, I'm not quoting its exact value, since it does not effect the argument of the exponential, where all the action is. Pressing on, after factoring out an overall phase, the probability becomes:

$$\mathcal{P}_{\ell\to\ell'} = \frac{g_S^2 g_D^2}{\pi^2 L^2} \left(2\sqrt{2\pi}\Delta x\right)^3 \int \frac{d^3\mathbf{p}_X}{(2\pi)^3}\, \bar{\rho}(\Delta_j, \bar{E}_*) \left|\sum_a U_{\ell a}^* U_{a\ell'} e^{-i\bar{\tau}E_*^a + ip_*^a L - \Delta x^2 (p_*^a \hat{z} + \mathbf{p}_X - \gamma M \mathbf{v})^2}\right|^2 \tag{49}$$

where $p_*^a \equiv \sqrt{(E_*^a)^2 - m_a^2}$, $\bar{\tau} \equiv \tau + T/2$, and

$$\bar{\rho}(\Delta_j, \bar{E}_*) \equiv \int d\Delta_j\ \rho(\Delta_j) \frac{e^{-\Gamma\tau}}{(\Delta_j - \bar{E}_*)^2 + \Gamma_D^2/4} \tag{50}$$

What remains is the $\mathbf{p}_X$ integral.

Stationary Source

For now, I'll set $\mathbf{v} = 0$, and do the calculation again later for nonzero $\mathbf{v}$. Let's change variables from a Cartesian coordinate system for $\mathbf{p}_X$ to one in spherical coordinates:

$$p_X^x \equiv k \sin\theta \cos\phi \tag{51}$$
$$p_X^y \equiv k \sin\theta \sin\phi \tag{52}$$
$$p_X^z \equiv k \cos\theta \tag{53}$$

Only the Gaussian in the integrand of Eq. (49) depends on the angular variables, since E_* and p_* only depend only on k^2. The terms in the squared amplitude will have the following Gaussian arguments in the exponential:

$$-\Delta x^2\left(p_*^a\hat{z}+\mathbf{p}_X\right)^2-\Delta x^2\left(p_*^b\hat{z}+\mathbf{p}_X\right)^2$$

$$=-2\Delta x^2(p_X^x)^2-2\Delta x^2(p_X^y)^2-\Delta x^2\left(p_*^a+p_X^z\right)^2-\Delta x^2\left(p_*^b+p_X^z\right)^2 \tag{54}$$

$$=-2\Delta x^2k^2\sin^2\theta-\Delta x^2\left(p_*^a+k\cos\theta\right)^2-\Delta x^2\left(p_*^b+k\cos\theta\right)^2 \tag{55}$$

$$=-\Delta x^2\left(2k^2\sin^2\theta+(p_*^a)^2+(p_*^b)^2+2k^2\cos^2\theta+2(p_*^a+p_*^b)k\cos\theta\right) \tag{56}$$

$$=-\Delta x^2\left(2k^2+(p_*^a)^2+(p_*^b)^2+2(p_*^a+p_*^b)k\cos\theta\right) \tag{57}$$

Letting $x=\cos\theta$, and isolating the angular integrals:

$$\int_{-1}^{1}dx\int_0^{2\pi}d\phi\; e^{-\Delta x^2\left(2k^2+(p_*^a)^2+(p_*^b)^2+2(p_*^a+p_*^b)kx\right)} \tag{58}$$

$$=2\pi e^{-\Delta x^2\left(2k^2+(p_*^a)^2+(p_*^b)^2\right)}\int_{-1}^{1}dx\; e^{-2\Delta x^2(p_*^a+p_*^b)kx} \tag{59}$$

$$=2\pi e^{-\Delta x^2\left(2k^2+(p_*^a)^2+(p_*^b)^2\right)}\;\frac{e^{2\Delta x^2(p_*^a+p_*^b)k)}}{2\Delta x^2(p_*^a+p_*^b)k}+\mathcal{O}\left(\frac{1}{\Delta x^4}\right) \tag{60}$$

$$=\frac{\pi\; e^{-\Delta x^2(k-p_*^a)^2-\Delta x^2\left(k-p_*^b\right)^2}}{\Delta x^2(p_*^a+p_*^b)k} \tag{61}$$

where again $p_*^a\equiv\sqrt{(E_*^a)^2-m_a^2}$. The above integral approximation is just the statement that the X state goes in the $-\hat{z}$ direction when $\mathbf{v}=0$. The probability is then:

$$\begin{aligned}\mathcal{P}_{\ell\to\ell'} &= \frac{g_S^2g_D^2}{\pi^3L^2}\left(2\sqrt{2\pi}\Delta x\right)\int_0^\infty kdk\;\bar{\rho}(\Delta_j,\bar{E}_*)\\ &\times\sum_{a,b}U_{\ell a}^*U_{a\ell'}U_{\ell b}U_{b\ell'}^*\;\frac{e^{-i\bar{\tau}(E_*^a-E_*^b)+iL(p_*^a-p_*^b)-\Delta x^2(k-p_*^a)^2-\Delta x^2(k-p_*^b)^2}}{(p_*^a+p_*^b)}\end{aligned} \tag{62}$$

Now, since E_*^a is defined as the root of Eq. (41) (it is not the on-shell energy of the neutrino!), it the same for all neutrino species:

$$E_*^a \overset{\text{Eq. (41)}}{=} M-E_X \tag{63}$$

$$\overset{\text{def}}{=} M-\sqrt{k^2+M_X^2} \tag{64}$$

So, here when $\mathbf{v}=0$, the $\bar{\tau}$ dependence in the oscillation probability drops out:

$$\begin{aligned}\mathcal{P}_{\ell\to\ell'} &= \frac{g_S^2g_D^2}{\pi^3L^2}\left(2\sqrt{2\pi}\Delta x\right)\int_0^\infty kdk\;\bar{\rho}(\Delta_j,\bar{E}_*)\\ &\times\sum_{a,b}U_{\ell a}^*U_{a\ell'}U_{\ell b}U_{b\ell'}^*\;\frac{e^{iL(p_*^a-p_*^b)-\Delta x^2(k-p_*^a)^2-\Delta x^2(k-p_*^b)^2}}{(p_*^a+p_*^b)}\end{aligned} \tag{65}$$

Here,

$$p_*^a \overset{\text{def}}{=} \sqrt{(E_*^a)^2-m_a^2} \tag{66}$$

$$\overset{\text{Eq. (41)}}{=} \sqrt{(M-E_X)^2-m_a^2} \tag{67}$$

$$\overset{\text{def}}{=} \sqrt{\left(M-\sqrt{k^2+M_X^2}\right)^2-m_a^2} \tag{68}$$

The remaining integral over k will be dominated by the saddle point. We have two Gaussians. Let's simplify one at a time, then combine them into a single Gaussian. For one of the Gaussians, we'll define k_*^a to be the value of k such that $k - p_*^a = 0$, which can be solved analytically:

$$k_*^a \equiv \frac{\sqrt{((M+M_X)^2 - m_a^2)((M-M_X)^2 - m_a^2)}}{2M} \tag{69}$$

(It's interesting to note that this is the same momentum one gets from solving the two-body on-shell kinematics for a stationary source.) Now, one can expand the argument in the exponential of the Gaussian to second order about $k = k_*^a$:

$$-\Delta x^2 (k - p_*^a)^2 = -\left(\frac{2M^2 \Delta x}{M^2 + M_X^2 - m_a^2}\right)^2 (k - k_*^a)^2 + \cdots \tag{70}$$

$$= -\left(\frac{2M^2 \Delta x}{M^2 + M_X^2}\right)^2 (k - k_*^a)^2 + \mathcal{O}\left(\frac{m_a^2}{M^2}\right) \tag{71}$$

to get the leading order behavior. Here, the ultrarelativistic approximation was used, so each Gaussian has a different saddle point, but their widths are the same, up to corrections of order m^2/M^2. Using this approximation, the two Gaussians can be combined into one by completing the square:

$$-\Delta x^2 (k - p_*^a)^2 - \Delta x^2 (k - p_*^b)^2 \sim -\left(\frac{2M^2 \Delta x}{M^2 + M_X^2}\right)^2 \left((k - k_*^a)^2 + (k - k_*^b)^2\right) \tag{72}$$

$$= -2\left(\frac{2M^2 \Delta x}{M^2 + M_X^2}\right)^2 \left(k - \frac{1}{2}(k_*^a + k_*^b)\right)^2 - \frac{1}{2}\left(\frac{2M^2 \Delta x}{M^2 + M_X^2}\right)^2 \left(k_*^a - k_*^b\right) \tag{73}$$

For the interference terms, the saddle point is located halfway in between the two Gaussians. Now, one can perform the remaining integral over k, in the limit that Δx is large enough to approximate the integral as only having a dominant contribution in a small neighborhood in k around $(k_*^a + k_*^b)/2$. If so, one can make the following replacements in the integrand of Eq. (65), using the ultra-relativistic approximation:

$$\left.(p_*^a - p_*^b)\right|_{k=(k_*^a + k_*^b)/2} = -\frac{(m_a^2 - m_b^2)}{2p_0} + \mathcal{O}(m^4) \tag{74}$$

$$\left.\frac{k}{p_*^a + p_*^b}\right|_{k=(k_*^a + k_*^b)/2} = \frac{1}{2} + \mathcal{O}(m^2) \tag{75}$$

$$e^{-\frac{1}{2}\left(\frac{2M^2 \Delta x}{M^2 + M_X^2}\right)^2 \left(k_*^a - k_*^b\right)} = 1 + \mathcal{O}((m_a^2 - m_b^2)\Delta x^2) \tag{76}$$

where

$$p_0 \equiv \frac{M^2 - M_X^2}{2M} \tag{77}$$

is equal to the on-shell energy of the neutrinos, if they were massless. Using these approximations in Eq. (65), we have the transition probability:

$$\mathcal{P}_{\ell \to \ell'} = \frac{g_S^2 g_D^2}{\pi^2 L^2} \left(\frac{M^2 + M_X^2}{2M^2}\right) \bar{\rho} \sum_{a,b} U_{\ell a}^* U_{a \ell'} U_{\ell b} U_{b \ell'}^* \, e^{-\frac{i(m_a^2 - m_b^2)L}{2p_0}} \tag{78}$$

Peeling off the oscillation probability:

$$\mathcal{P}^{\text{osc}}_{\nu_\ell \to \nu_{\ell'}} = \sum_{a,b} U^*_{\ell a} U_{a\ell'} U_{\ell b} U^*_{b\ell'} \; e^{-\frac{i(m_a^2 - m_b^2)L}{2p_0}} \tag{79}$$

$$= \left| \sum_a U^*_{\ell a} U_{a\ell'} \; e^{-\frac{i m_a^2 L}{2p_0}} \right|^2 \tag{80}$$

This is the standard neutrino oscillation formula used by experiments, for a stationary source. This relied on an additional assumption that $(m_a^2 - m_b^2)\Delta x^2 \ll 1$. Crazy how that happened.

Source with $\mathbf{v} = v\hat{z}$

Let's redo the probability calculation, but now with $\mathbf{v} = v\hat{z}$. This approximates what is done in pion beam experiments. Let's change variables from a Cartesian coordinate system for $\mathbf{p}_X$ to one in spherical coordinates:

$$p_X^x \equiv k \sin\theta \cos\phi \tag{81}$$

$$p_X^y \equiv k \sin\theta \sin\phi \tag{82}$$

$$p_X^z \equiv k \cos\theta \tag{83}$$

Now, the Gaussian in the integrand of Eq. (49) is not the only part of the integrand that depends on the angular variables, since E_* and p_* now depend on θ, as defined in Eq. (41). We're going to treat the Gaussians as the dominant contribution to the integral, since deviations from their center correspond with The terms in the squared amplitude will have the following Gaussian arguments in the exponential:

$$-\Delta x^2 \left(p_*^a \hat{z} + \mathbf{p}_X - \gamma M v \hat{z}\right)^2 - \Delta x^2 \left(p_*^b \hat{z} + \mathbf{p}_X - \gamma M v \hat{z}\right)^2 \tag{84}$$

$$= -2\Delta x^2 (p_X^x)^2 - 2\Delta x^2 (p_X^y)^2 - \Delta x^2 \left(p_*^a + p_X^z - \gamma M v\right)^2 - \Delta x^2 \left(p_*^b + p_X^z - \gamma M v\right)^2$$

$$= -2\Delta x^2 k^2 \sin^2\theta - \Delta x^2 \left(p_*^a + k\cos\theta - \gamma M v\right)^2 - \Delta x^2 \left(p_*^b + k\cos\theta - \gamma M v\right)^2 \tag{85}$$

$$= -\Delta x^2 \left(2k^2 + (p_*^a)^2 + (p_*^b)^2 - 2M\gamma v(p_*^a + p_*^b) + 2M^2\gamma^2 v^2 + 2(p_*^a + p_b^* - 2M\gamma v) k\cos\theta\right) \tag{86}$$

Letting $x = \cos\theta$, and isolating the angular integrals:

$$\int_{-1}^{1} dx \int_0^{2\pi} d\phi \; f(kx) \; e^{-\Delta x^2 \left(2k^2 + (p_*^a)^2 + (p_*^b)^2 - 2M\gamma v(p_*^a + p_*^b) + 2M^2\gamma^2 v^2 + 2(p_*^a + p_b^* - 2M\gamma v)kx\right)} \tag{87}$$

$$= 2\pi e^{-\Delta x^2 \left(2k^2 + (p_*^a)^2 + (p_*^b)^2 - 2M\gamma v(p_*^a + p_*^b) + 2M^2\gamma^2 v^2\right)} \int_{-1}^{1} dx \; f(kx) \; e^{-2\Delta x^2 (p_*^a + p_b^* - 2M\gamma v)kx} \tag{88}$$

$$= 2\pi e^{-\Delta x^2 \left(2k^2 + (p_*^a)^2 + (p_*^b)^2 - 2M\gamma v(p_*^a + p_*^b) + 2M^2\gamma^2 v^2\right)} \frac{f(\eta k) \; e^{2\Delta x^2 |p_*^a + p_b^* - 2M\gamma v| k}}{2\Delta x^2 |p_*^a + p_b^* - 2M\gamma v| k} + \mathcal{O}\left(\frac{1}{\Delta x^2}\right) \tag{89}$$

$$= \frac{f(\eta k) \pi e^{-\Delta x^2 (\eta k - p_*^a + \gamma M v)^2 - \Delta x^2 \left(\eta k - p_*^b + \gamma M v\right)^2}}{\Delta x^2 \eta (p_*^a + p_b^* - 2M\gamma v) k} \tag{90}$$

where η is an indicator, defined as

$$\eta \equiv \text{sign}\left(p_*^a + p_*^b - 2M\gamma v\right) \tag{91}$$

so $\eta^2 = 1$. The above integral approximation is just the statement that the X state goes in the $-\hat{z}$ direction when v is small enough, and in the $+\hat{z}$ direction when v is large enough. That is, the saddle point picks out $\mathbf{p}_X \sim -\eta k\hat{z} + \mathcal{O}(1/\Delta x)$. The probability is then:

$$\begin{aligned}\mathcal{P}_{\ell\to\ell'} &= \frac{g_S^2 g_D^2}{\pi^3 L^2}\left(2\sqrt{2\pi}\Delta x\right)\int_0^\infty kdk\ \bar{\rho}(\Delta_j, \bar{E}_*) \\ &\quad\times \sum_{a,b} U^*_{\ell a}U_{a\ell'}U_{\ell b}U^*_{b\ell'}\ \frac{e^{-i\bar{\tau}(E^a_*-E^b_*)+iL(p^a_*-p^b_*)-\Delta x^2(\eta k-p^a_*+\gamma Mv)^2-\Delta x^2\left(\eta k-p^b_*+\gamma Mv\right)^2}}{\eta(p^a_*+p^b_*-2M\gamma v)}\end{aligned} \tag{92}$$

Still $p^a_* \equiv \sqrt{(E^a_*)^2 - m_a^2}$, but now E^a_* is defined as the root of Eq. (41):

$$0 = E_X + E^a_* - M/\gamma - v\left(\sqrt{(E^a_*)^2 - m_a^2} - \eta k\right) \tag{93}$$

$$= \sqrt{k^2 + M_X^2} + E^a_* - M/\gamma - v\left(\sqrt{(E^a_*)^2 - m_a^2} - \eta k\right) \tag{94}$$

At this point, one can analytically solve the above equation for E^a_*, in terms of k. However, it's a bit cumbersome. Importantly, the value of E^a_* no longer is independent of m_a, so the $\bar{\tau}$ dependence in the oscillation probability does not drop out, like it did in the $\mathbf{v} = 0$ case. It's going to be more convenient to change variables from E^a_* to $p^a_* = \sqrt{(E^a_*)^2 - m_a^2}$ in the above equation:

$$0 = \sqrt{k^2 + M_X^2} + \sqrt{(p^a_*)^2 + m_a^2} - M/\gamma - v\left(p^a_* - \eta k\right) \tag{95}$$

The remaining integral over k will be dominated by the saddle point. We have two Gaussians. Let's simplify one at a time, then combine them into a single Gaussian. For one of the Gaussians, we'll define k^a_* to be the value of k such that $\eta k - p^a_* + \gamma Mv = 0$, where p^a_* is the root of Eq. (95). This is not easy to solve directly. Instead, it can be solved by writing down the correct answer:

$$\eta k^a_* \equiv \frac{\gamma\sqrt{((M+M_X)^2 - m_a^2)((M-M_X)^2 - m_a^2)}}{2M} - \frac{\gamma v(M^2 + M_X^2 - m_a^2)}{2M} \tag{96}$$

and then verifying it is correct by plugging this value of k into $0 = (\eta k - p^a_* + \gamma Mv)\big|_{k=k^a_*}$. One will recognize this expression of k^a_* as the magnitude of the on-shell momentum of the X state, in the boosted frame with velocity v. Expanding the Gaussian about k^a_* to second order:

$$-\Delta x^2(\eta k - p^a_* + \gamma Mv)^2 = -\Delta x^2 g(M, M_X, v)(k - \eta k^a_*)^2 + \mathcal{O}\left(\frac{m_a^2}{M^2}\right) \tag{97}$$

where g is some function I'm not going to waste anymore time trying to solve for. The point is that the saddle point is at $k = k^a_*$, and the exact width of the Gaussian doesn't matter in the end, since it doesn't change the pole structure. The two Gaussians in Eq. (92) can be combined into one:

$$\begin{aligned}&-\Delta x^2(\eta k - p^a_* + \gamma Mv)^2 - \Delta x^2(\eta k - p^b_* + \gamma Mv)^2 \\ &\quad\sim -\Delta x^2 g(M, M_X, v)\Big((k - \eta k^a_*)^2 + (k - \eta k^b_*)^2\Big) \qquad (98)\\ &\quad= -2\Delta x^2 g(M, M_X, v)\Big(k - \frac{1}{2}(k^a_* + k^b_*)\Big)^2 - \frac{1}{2}\Delta x^2 g(M, M_X, v)\Big(k^a_* - k^b_*\Big) \qquad (99)\end{aligned}$$

Now that we have a single Gaussian, the integral can be solved using the saddle point approximation. The following terms in the integrand can be approximated as the following, in the limit that m_a is very small:

$$(E_*^a - E_*^b)\Big|_{k=(k_*^a+k_*^b)/2} = -\frac{\gamma v(m_a^2 - m_b^2)}{2p_0} + \mathcal{O}(m^4) \tag{100}$$

$$(p_*^a - p_*^b)\Big|_{k=(k_*^a+k_*^b)/2} = -\frac{\gamma(m_a^2 - m_b^2)}{2p_0} + \mathcal{O}(m^4) \tag{101}$$

$$\frac{k}{\eta(p_*^a + p_*^b - 2M\gamma v)}\Bigg|_{k=(k_*^a+k_*^b)/2} = \frac{1}{2} + \mathcal{O}(m^2) \tag{102}$$

$$e^{-\frac{1}{2}\Delta x^2 g(M,M_X,v)\left(k_*^a-k_*^b\right)} = 1 + \mathcal{O}\left((m_a^2 - m_b^2)\Delta x^2\right) \tag{103}$$

where p_0 is defined as before:

$$p_0 \equiv \frac{M^2 - M_X^2}{2M} \tag{104}$$

So, finally doing the Gaussian integral over k:

$$\mathcal{P}_{\ell\to\ell'} = \frac{g_S^2 g_D^2}{\pi^2 L^2}\frac{2}{\sqrt{g(M, M_X, v)}}\bar{\rho}\sum_{a,b} U_{\ell a}^* U_{a\ell'} U_{\ell b} U_{b\ell'}^* \; e^{-\frac{i\gamma(m_a^2-m_b^2)}{2p_0}(L-\bar{\tau}v)} \tag{105}$$

This is the main result.

Setting $\bar{\tau} \simeq L$, the probability becomes:

$$\mathcal{P}_{\ell\to\ell'} = \frac{g_S^2 g_D^2}{\pi^2 L^2}\frac{2}{\sqrt{g(M, M_X, v)}}\bar{\rho}\sum_{a,b} U_{\ell a}^* U_{a\ell'} U_{\ell b} U_{b\ell'}^* \; e^{-\frac{i(m_a^2-m_b^2)L}{2p_0'}} \tag{106}$$

where

$$p_0' \equiv \frac{p_0}{\gamma(1-v)} \tag{107}$$

$$= \frac{M^2 - M_X^2}{2M\gamma(1-v)} \tag{108}$$

which is exactly the momentum of the neutrino in the lab frame, if the parents particle has a velocity $\mathbf{v} = v\hat{z}$. The three-body kinematics is a result of this calculation, not an direct input. Peeling off the oscillation probability:

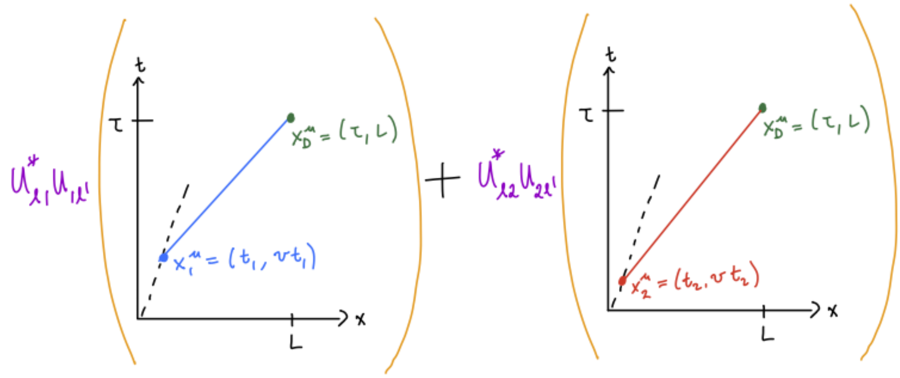
$$\mathcal{P}^{\text{osc}}_{\nu_\ell\to\nu_{\ell'}} = \sum_{a,b} U_{\ell a}^* U_{a\ell'} U_{\ell b} U_{b\ell'}^* \; e^{-\frac{i(m_a^2-m_b^2)L}{2p_0'}} \tag{109}$$

$$= \left|\sum_a U_{\ell a}^* U_{a\ell'} \; e^{-\frac{im_a^2 L}{2p_0'}}\right|^2 \tag{110}$$

This is the standard neutrino oscillation formula used by experiments, for a moving source. This relied on an additional assumption that $(m_a^2 - m_b^2)\Delta x^2 \ll 1$.

Geometric Semi-Classical Interpretation

This semi-classical interpretation reproduces the τ dependence in the phase in Eq. (105). The figure below shows the addition of two amplitudes associated with neutrino propagation from the source to the detector, one with neutrino ν_1 and one with neutrino ν_2. The source has a mass M, begins at the origin in spacetime, and travels in the lab frame with velocity v, in the direction of the detector. The source decays to two sets of particles: one neutrino with mass m_i ($i = 1, 2$), and a set of other particles with invariant mass M_X. The point in spacetime along the source's worldline where the neutrino and the X are emitted is $x_i^\mu = (t_i, vt_i)$. The neutrinos emitted then travel in a straight trajectory in space time on their way to the detector located at $x_D^\mu = (\tau, L)$ in the lab frame.



The neutrino oscillation amplitude is

$$\mathcal{A}^{\text{osc}}_{\nu_\ell \to \nu_{\ell'}} = \sum_i U^*_{\ell i} U_{i\ell'} e^{i\phi_i} \tag{111}$$

where the phase for each diagram ϕ_i has a contribution from the source, the X state, and the neutrino:

$$\phi_i = -p_S \cdot x_i + p_X \cdot x_i - p_i \cdot (x_D - x_i) \tag{112}$$

The sign convention is that the spacetime translation phase gets a minus (plus) sign if it's going into (coming out of) a vertex. Because of energy-momentum conservation: $p_S^\mu = p_X^\mu + p_i^\mu$, the expression for ϕ_i can be simplified:

$$\phi_i = -p_i \cdot x_D \tag{113}$$

This is not to be interpreted that all neutrinos travel the same distance in space time. Rather, it's a consequence of the linear combination of phases and energy-momentum conservation. Since τ and L are defined in the lab frame, where the source particle has velocity v, we have to express the 4-momentum of the neutrino in the lab frame as well:

$$\begin{aligned} p_i^\mu &= (E_i', p_i') \qquad &(114)\\ &= \left(\gamma E + \gamma v\sqrt{E^2 - m_i^2}, \gamma\sqrt{E^2 - m_i^2} + \gamma v E\right) \qquad &(115) \end{aligned}$$

Here's the point to glean from the QFT calculation: the neutrinos that interfere have the same energy E in the CM frame of the source. This is because the detector in calculation is a heavy 2-state system, which measures energy. The value of E is the mean between the two on-shell energies of the neutrinos. This average comes about because when the diagram interfere, they each have a saddle point centered at on-shell momentum, so when the diagrams are combined, the saddle points combine into one, exactly halfway in between. The resulting single saddle point usually would be a source of decoherence, but since the neutrino masses so much lighter than all of energy scales in the problem, this decoherence is not sizable. So, the phase in the interference terms have the form:

$$\begin{aligned}
\phi_1 - \phi_2 &= -(E_1' - E_2')\tau + (p_1' - p_2')L && (116)\\
&= -\gamma v \left(\sqrt{E^2 - m_1^2} - \sqrt{E^2 - m_2^2}\right)\tau + \gamma\left(\sqrt{E^2 - m_1^2} - \sqrt{E^2 - m_2^2}\right) L && (117)\\
&= \gamma\left(\sqrt{E^2 - m_1^2} - \sqrt{E^2 - m_2^2}\right)(L - \tau v) && (118)
\end{aligned}$$

In the CM frame of the source, the value of E is

$$E = \frac{1}{2}\left(\frac{M^2 - M_X^2 - m_1^2}{2M} + \frac{M^2 - M_X^2 - m_2^2}{2M}\right) \quad (119)$$

Expanding to order m^2, we have:

$$\begin{aligned}
\phi_1 - \phi_2 &= \gamma M \left(\frac{m_1^2 - m_2^2}{M^2 + M_X^2}\right)(L - \tau v) + \mathcal{O}(m^4) && (120)\\
&= \frac{\gamma(m_1^2 - m_2^2)}{2p_0}(L - \tau v) && (121)
\end{aligned}$$

where

$$p_0 = \frac{M^2 - M_X^2}{2M} \quad (122)$$

is the energy (or momentum) of the neutrino in the CM frame of the source, if the neutrino were massless. Therefore, the oscillation probability is

$$P^{\rm osc}_{\nu_\ell \to \nu_{\ell'}} = \left|\sum_i U^*_{\ell i} U_{i\ell'} e^{\frac{i\gamma m_i^2 (L - \tau v)}{2p_0}}\right|^2 \quad (123)$$

This reproduces the QFT oscillation probability in Eq. (105).

Now, what is the $L - \tau v$ factor doing? It changes the effective baseline. To see how in this geometric interpretation, say the source with velocity v in the lab frame travels a distance x before emitting the neutrino. Since the neutrino travels in a straight line:

$$\tau = \frac{x}{v} + (L - x) + \mathcal{O}(m^2) \quad (124)$$

and so,

$$L - v\tau = (1 - v)(L - x) + \mathcal{O}(m^2) \quad (125)$$

The value of $L - x$ is the effective distance that the neutrino travels from being emitted to the detector. Therefore, the phase in Eq. (123) becomes:

$$P^{\rm osc}_{\nu_\ell \to \nu_{\ell'}} = \left|\sum_i U^*_{\ell i} U_{i\ell'} e^{\frac{i m_i^2 \tilde{L}}{2p_0'}}\right|^2 \quad (126)$$

where $\tilde{L} \equiv L - x$ is the effective baseline of the oscillation experiment, and

$$p_0' \equiv \frac{M^2 - M_X^2}{2M\gamma(1 - v)} \quad (127)$$

is the on-shell energy of the neutrino in the lab frame, if it were massless.