Notes on the Non-Exponential Tail of Particle Decay

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Contents

These notes explore the non-exponential tail in the survival probability of unstable particles, as discussed in Supplement II of Sakurai's *Modern Quantum Mechanics* [1]. (Note that the topic of non-exponential particle decay is absent from the 3rd edition of the textbook [2].) Sakurai's discussion draws largely from Khalfin's 1958 paper [3].

The hope here is to try to calculate the survival probability directly from a general microscopic theory, which will show how the analytic structure arises from a resumming a perturbative interaction, instead of simply positing the structure of the spectral density. Then the Breit-Wigner approximation will be reviewed, and Khalfin's derivation of the non-exponential long-time behavior of the survival amplitude will be recreated. Finally, there is a discussion on how the lowest terms in a threshold expansion can be shown to dominant the functional form of the survival probability as $t \to \infty$.

1 Resumming a Perturbative Calculation from Scratch

Initial state. To begin, assume you know there is only a single particle P with mass M in your lab at time t_0, where the length scale of its wavefunction Δx is parametrically larger than it Compton wavelength $1/M$. This implies this is a non-relativistic system, and such a state can be described by the single-particle quantum state:

$$|\psi(t=0)\rangle = \int d^d\mathbf{x}\ \psi(\mathbf{x})\,|\mathbf{x}\rangle\,, \tag{1}$$

where d is the number of spacial dimensions. Here, $\psi(\mathbf{x})$ can be thought of as the square-root of a Gaussian with spatial support of scale Δx, where $\int d^d\mathbf{x}\ |\psi(\mathbf{x})|^2 = 1$. In the non-relativistic limit the states $|\mathbf{x}\rangle$ have the following momentum state normalizations:

$$|\mathbf{x}\rangle = \int \frac{d^d\mathbf{k}}{(2\pi)^d}\ e^{i\mathbf{k}\cdot\mathbf{x}}\,|\mathbf{k}\rangle\,, \qquad |\mathbf{k}\rangle = \int d^d\mathbf{x}\ e^{-i\mathbf{k}\cdot\mathbf{x}}\,|\mathbf{x}\rangle\,. \tag{2}$$

Then there also admits the momentum state representation

$$|\psi(t=0)\rangle = \int \frac{d^d\mathbf{k}}{(2\pi)^d}\ \widetilde{\psi}(\mathbf{k})\,|\mathbf{k}\rangle\,, \tag{3}$$

where $\widetilde{\psi}(\mathbf{k})$ is the Fourier transform of $\psi(\mathbf{x})$.

Consider the time scale of the experiment is time T. In the limit $T/(M\Delta x^2) \to 0$, the free diffusion of the particle spatial wave function can be ignored for the duration of the experiment. In this $M \to \infty$ limit, the particle has momentum $\mathbf{p} = M\mathbf{v} + \mathbf{k}$, and we can take its velocity $\mathbf{v} = \mathbf{0}$. (For details on this non-relativistic approximation, see Ref. [4]. This is simply HQET in spacetime in the $M \to \infty$ limit.) Later on, we will consider the limit where the wavefunction is sharply peaked at zero momentum:

$$|\widetilde{\psi}(\mathbf{k})|^2 \sim (2\pi)^d\delta^d(\mathbf{k}). \tag{4}$$

Time evolution pictures. When considering later times, we have to pay attention to the differences phases between the Schrödinger picture and the interaction picture. So far, we've only considered the state at $t = 0$, where there are no interactions, so there is no difference between the Schrödinger picture and the interaction picture. Here,

$$\hat{H}\,|\psi(t=0)\rangle = \hat{H}_0\,|\psi(t=0)\rangle = M\,|\psi(t=0)\rangle\,. \tag{5}$$

But for $t > 0$, we have to be more careful with the pictures so the overall phase comes out correct. In the Schrödinger picture, the particle state at a later time t is:

$$|\psi_S(t)\rangle = e^{-i\hat{H}t}\,|\psi_S(t=0)\rangle\,, \tag{6}$$

where H is the full Hamiltonian $H = H_0 + V(t)$, where V contains the interaction operators, which in general depends on time. In the interaction picture, we have the states

$$|\psi_I(t)\rangle = e^{i\hat{H}_0 t}\,|\psi_S(t)\rangle = e^{i\hat{H}_0 t}e^{-i\hat{H}t}\,|\psi_S(t=0)\rangle\,. \tag{7}$$

Interactions. Consider an interaction that allows the particle to decay to a collection of particles X:

$$V(t) = \int d^d\mathbf{x}\ \mathcal{O}_{P\to X}(t,\mathbf{x}) + H.c., \tag{8}$$

where $\mathcal{O}_{P\to X}(t,\mathbf{x})$ is a local spacetime operator that mediates that transition from the parent particle P to the daughter state X, and "*H.c.*" denotes the Hermitian conjugate. The local operator can be translated in spacetime like so:

$$\mathcal{O}_{P\to X}(t,\mathbf{x}) = e^{i\hat{H}_0 t - i\hat{\mathbf{P}}\cdot\mathbf{x}}\mathcal{O}_{P\to X}(0,\mathbf{0})e^{-i\hat{H}_0 t + i\hat{\mathbf{P}}\cdot\mathbf{x}}. \tag{9}$$

Here, the only nonzero matrix element for $\mathcal{O}_{P\to X}$ are ones that transition between the parent and the daughter states:

$$\langle X|\, \mathcal{O}_{P\to X}(t,\mathbf{x})\, |\psi(t)\rangle \neq 0, \tag{10}$$

or, more explicitly, using Eqs. (3) and (9):

$$\begin{aligned} \langle X|\, \mathcal{O}_{P\to X}(t,\mathbf{x})\, |\psi(t)\rangle &= \int \frac{d^d\mathbf{k}}{(2\pi)^d}\, \widetilde{\psi}(\mathbf{k})\, \langle X|\, \mathcal{O}_{P\to X}(t,\mathbf{x})\, |\mathbf{k}\rangle\,, & (11)\\ &= \int \frac{d^d\mathbf{k}}{(2\pi)^d}\, \widetilde{\psi}(\mathbf{k}) e^{iE_X t - i\mathbf{p}_X\cdot\mathbf{x} - iMt + i\mathbf{k}\cdot\mathbf{x}}\, \langle X|\, \mathcal{O}_{P\to X}(0,\mathbf{0})\, |\mathbf{k}\rangle\,. & (12)\end{aligned}$$

Letting $\langle X|\, \mathcal{O}_{P\to X}(0,\mathbf{0})\, |\mathbf{k}\rangle = F(X;\mathbf{p}_X,\mathbf{k})$, we can write Eq. (12) as:

$$\langle X|\, \mathcal{O}_{P\to X}(t,\mathbf{x})\, |\psi(t)\rangle = \int \frac{d^d\mathbf{k}}{(2\pi)^d}\, \widetilde{\psi}(\mathbf{k}) e^{i(E_X - M)t - i(\mathbf{p}_X - \mathbf{k})\cdot\mathbf{x}} F(X;\mathbf{p}_X,\mathbf{k}). \tag{13}$$

The notation $F(X;\mathbf{p}_X,\mathbf{k})$ is meant to suggest that the this function depends not only on $\mathbf{p}_X$ and $\mathbf{k}$, but also on the configuration of the set of particle that constitute the state X.

Daughter states. The final state particles X can be taken to be external scattering states. Given this, the final state X can be represented as a state $|X\rangle$, which is an eigenstate of momentum and the free Hamiltonian:

$$\hat{\mathbf{P}}\, |X\rangle = \mathbf{p}_X\, |X\rangle\,, \qquad H_0\, |X\rangle = E_X\, |X\rangle\,, \tag{14}$$

where $E_X = \sqrt{M_X^2 + \mathbf{p}_X^2}$, where M_X is the invariant mass of the X system. Again, note that X is not a single particle, but represents a system of on-shell states with well-defined momentum and energy. Energy and momentum conservation between the parent state the X state will be enforced in the upcoming integrals.

Survival amplitude. The physical survival amplitude is

$$\begin{aligned} A(t) &= \langle \psi_S(t=0)|\psi_S(t)\rangle\,, & (15)\\ &= \langle \psi_S(t=0)|\, e^{-i\hat{H}t}\, |\psi_S(t=0)\rangle\,, & (16)\\ &= \langle \psi_I(t=0)|\, e^{i\hat{H}_0 t} e^{-i\hat{H}t} e^{-i\hat{H}_0 t}\, |\psi_I(t=0)\rangle\,, & (17)\\ &= \langle \psi_I(t=0)|\, e^{-i\hat{H}t}\, |\psi_I(t=0)\rangle\,, & (18)\\ &= \langle \psi_I(t=0)|\, e^{-i\hat{H}_0 t} U(t)\, |\psi_I(t=0)\rangle\,, & (19)\\ &= e^{-iMt}\, \langle \psi_I(t=0)|\, U(t)\, |\psi_I(t=0)\rangle\,, & (20)\\ &\equiv e^{-iMt} a(t). & (21)\end{aligned}$$

We will use time-dependent perturbation theory in the interaction picture to compute this matrix element:

$$a(t) = \langle \psi_I(t=0)|\, U(t)\, |\psi_I(t=0)\rangle\,, \tag{22}$$

where the probability that the particle survives a time t is $P(t) = |a(t)|^2$. The time evolution operator with interactions is:

$$U(t) = \mathcal{T} e^{-i\int_0^t V(t')dt'}, \tag{23}$$

where $\mathcal{T}$ represents the time-ordered product, and $V(t)$ is the interaction defined in Eq. (8). Let $|\psi_I(t=0)\rangle = |\psi\rangle$ for brevity. Taylor expanding $U(t)$, we have (ignoring the irrelevant overall phase e^{iMt}):

$$\begin{aligned} a(t) &= 1 + (-i)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \langle\psi| V(t_1)V(t_2) |\psi\rangle \\ &\quad + (-i)^4 \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 \langle\psi| V(t_1)V(t_2)V(t_3)V(t_4) |\psi\rangle \\ &\quad + \cdots \end{aligned} \tag{24}$$

Only the terms with an even number of V's survive. We'll label each of these terms the following:

$$a(t) = 1 + a^{(2)}(t) + a^{(4)}(t) + \cdots \tag{25}$$

1st order. We will start by calculating the 1st-order correction to the survival amplitude:

$$a^{(2)}(t) = (-i)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \langle\psi| V(t_1)V(t_2) |\psi\rangle . \tag{26}$$

Inserting a resolution of the identity over all states in the Hilbert space:

$$\mathbb{I} = |\psi\rangle\langle\psi| + \sum_X \int \prod_{j\in X} \frac{d^d\mathbf{p}_j}{(2\pi)^d 2E_j} |X\rangle\langle X| , \tag{27}$$

and picking out only those X states that have nonzero matrix element according to Eq. (10):

$$a^{(2)}(t) = -\int_0^t dt_1 \int_0^{t_1} dt_2 \sum_X \int \prod_{j\in X} \frac{d^d\mathbf{p}_j}{(2\pi)^d 2E_j} \langle\psi| V(t_1) |X\rangle \langle X| V(t_2) |\psi\rangle . \tag{28}$$

Using Eq. (8):

$$\begin{aligned} a^{(2)}(t) &= -\int_0^t dt_1 \int_0^{t_1} dt_2 \sum_X \int \prod_{j\in X} \frac{d^d\mathbf{p}_j}{(2\pi)^d 2E_j} \int d^d\mathbf{x} \int d^d\mathbf{y} \\ &\quad \times \langle\psi| \mathcal{O}^\dagger_{X\to P}(t_1,\mathbf{x}) |X\rangle \langle X| \mathcal{O}_{P\to X}(t_2,\mathbf{y}) |\psi\rangle . \end{aligned} \tag{29}$$

Using Eq. (13):

$$\begin{aligned} a^{(2)}(t) &= -\int_0^t dt_1 \int_0^{t_1} dt_2 \sum_X \int \prod_{j\in X} \frac{d^d\mathbf{p}_j}{(2\pi)^d 2E_j} \int d^d\mathbf{x} \int d^d\mathbf{y} \int \frac{d^d\mathbf{k}}{(2\pi)^d} \int \frac{d^d\mathbf{q}}{(2\pi)^d} \\ &\quad \times e^{iE_X t_2 - i\mathbf{p}_X\cdot\mathbf{y} - iMt_2 + i\mathbf{k}\cdot\mathbf{y} - iE_X t_1 + i\mathbf{p}_X\cdot\mathbf{x} + iMt_1 - i\mathbf{q}\cdot\mathbf{x}} \\ &\quad \times \widetilde{\psi}^*(\mathbf{q})\widetilde{\psi}(\mathbf{k}) F^*(X;\mathbf{p}_X,\mathbf{q}) F(X;\mathbf{p}_X,\mathbf{k}). \end{aligned} \tag{30}$$

Doing the integrals over $\mathbf{x}$ and $\mathbf{y}$:

$$\begin{aligned} a^{(2)}(t) &= -\int_0^t dt_1 \int_0^{t_1} dt_2 \sum_X \int \prod_{j\in X} \frac{d^d\mathbf{p}_j}{(2\pi)^d 2E_j} \int \frac{d^d\mathbf{k}}{(2\pi)^d} \int \frac{d^d\mathbf{q}}{(2\pi)^d} \\ &\quad \times e^{iE_X(t_2-t_1) - iM(t_2-t_1)} (2\pi)^d \delta^d(\mathbf{q}-\mathbf{p}_X)(2\pi)^d \delta^d(\mathbf{k}-\mathbf{p}_X) \\ &\quad \times \widetilde{\psi}^*(\mathbf{q})\widetilde{\psi}(\mathbf{k}) F^*(X;\mathbf{p}_X,\mathbf{q}) F(X;\mathbf{p}_X,\mathbf{k}). \end{aligned} \tag{31}$$

Doing the integrals over $\mathbf{k}$ and $\mathbf{q}$, letting $F(X) = F(X; \mathbf{p}_X, \mathbf{p}_X)$:

$$\begin{aligned} a^{(2)}(t) &= -\int_0^t dt_1 \int_0^{t_1} dt_2 \sum_X \int \prod_{j\in X} \frac{d^d\mathbf{p}_j}{(2\pi)^d 2E_j} \\ &\quad \times e^{i(E_X-M)(t_2-t_1)} |\widetilde{\psi}(\mathbf{p}_X)|^2 |F(X)|^2. \end{aligned} \tag{32}$$

Using Eq. (4):

$$\begin{aligned} a^{(2)}(t) &= -\int_0^t dt_1 \int_0^{t_1} dt_2 \sum_X \int \prod_{j\in X} \frac{d^d\mathbf{p}_j}{(2\pi)^d 2E_j} \\ &\quad \times e^{i(E_X-M)(t_2-t_1)} (2\pi)^d \delta^d(\mathbf{p}_X) |F(X)|^2. \end{aligned} \tag{33}$$

The factor of $\delta^d(\mathbf{p}_X)$ enforces that the integral over the daughter particle momenta sum to zero, since the parent particle's momentum was taken to be zero. Now we can note the identity:

$$1 = \int_0^\infty \frac{dE}{2\pi}\ (2\pi)\delta(E - E_X). \tag{34}$$

which is true when $E_X > 0$. Using this, can write Eq. (33) as:

$$\begin{aligned} a^{(2)}(t) &= -\int_0^\infty dE \int_0^t dt_1 \int_0^{t_1} dt_2 \sum_X \int \prod_{j\in X} \frac{d^d\mathbf{p}_j}{(2\pi)^d 2E_j} \\ &\quad \times e^{i(E-M)(t_2-t_1)} (2\pi)^d \delta(E - E_X) \delta^d(\mathbf{p}_X) |F(X)|^2. \end{aligned} \tag{35}$$

Define the spectral density:

$$\rho(E) = \frac{1}{2\pi} \sum_X \int \prod_{j\in X} \frac{d^d\mathbf{p}_j}{(2\pi)^d 2E_j} (2\pi)^{d+1} \delta(E - E_X) \delta^d(\mathbf{p}_X) |F(X)|^2 \theta(E - E_X), \tag{36}$$

where θ is the Heaviside function. While using the identity in Eq. (34) may seem a bit slick, this step yields nearly the same expression for the spectral density found in the derivation of the Lehmann-Källén spectral representation of the two-point wavefunction, e.g., see Section 24.2.1 of Ref. [5]. Note that if the lowest-energy X state has a threshold energy E_{th}, then $\rho(E) \geq 0$ for $E \geq E_{th}$, and $\rho(E) = 0$ for $E \leq E_{th}$. Importantly, $E_{th} < M$.

Now Eq. (35) becomes:

$$a^{(2)}(t) = -\int_0^\infty dE \int_0^t dt_1 \int_0^{t_1} dt_2\ e^{-i(E-M)(t_1-t_2)} \rho(E). \tag{37}$$

Define $K(\tau)$:

$$K(\tau) = \theta(\tau) \int_0^\infty dE\ e^{-i(E-M)\tau} \rho(E). \tag{38}$$

Eq. (37) becomes

$$a^{(2)}(t) = -\int_0^t dt_1 \int_0^{t_1} dt_2\ K(t_1 - t_2), \tag{39}$$

i.e., one can make the identification:

$$K(t_1 - t_2) = \langle\psi| V(t_1)V(t_2) |\psi\rangle \,. \tag{40}$$

Because the integrand of Eq. (39) only depends on $t_1 - t_2$, we can further simplify the integral by letting $\tau = t_1 - t_2$ and changing the order of integration and performing one of the time integrals:

$$\begin{aligned}
a^{(2)}(t) &= -\int_0^t dt_1 \int_0^{t_1} d\tau\ K(\tau), && (41)\\
&= -\int_0^t d\tau \int_\tau^{t} dt_1\ K(\tau), && (42)\\
&= -\int_0^t d\tau\ (t-\tau)K(\tau). && (43)
\end{aligned}$$

Later on, we will perform a resummation of the Dyson series to all orders. To do so, we will need the Laplace transform:

$$\begin{aligned}
\widetilde{a}^{(2)}(s) &= \int_0^\infty dt\ e^{-st}a^{(2)}(t), && (44)\\
&= -\int_0^\infty dt\ e^{-st} \int_0^t d\tau\ (t-\tau)K(\tau), && (45)\\
&= -\int_0^\infty d\tau\ K(\tau) \int_\tau^\infty dt\ (t-\tau)e^{-st}, && (46)\\
&= -\frac{1}{s^2}\int_0^\infty d\tau\ e^{-s\tau}K(\tau), && (47)\\
&= -\frac{1}{s^2}\widetilde{K}(s), && (48)
\end{aligned}$$

where $\widetilde{K}(s)$ is the Laplace tranform of $K(\tau)$:

$$\begin{aligned}
\widetilde{K}(s) &= \int_0^\infty d\tau\ e^{-s\tau}K(\tau), && (49)\\
&= \int_0^\infty d\tau\ e^{-s\tau} \int_0^\infty dE\ e^{-i(E-M)\tau}\rho(E), && (50)\\
&= \int_0^\infty dE\ \frac{\rho(E)}{s + i(E-M)} && (51)
\end{aligned}$$

Since $\rho(E)$ is zero for $E < E_{th}$, according to Eq. (36), we can rewrite this as:

$$\widetilde{K}(s) = \int_{E_{th}}^\infty dE\ \frac{\rho(E)}{s + i(E-M)}. \tag{52}$$

Why the Laplace transform and not the Fourier transform? This is because $a(t)$ is not just 0 for $t < 0$, it is not defined for $t < 0$. The Laplace transforms allows for a mathematically clean way to handle problem which does not extend beyond the domain of applicability and it ensures convergence with the exponential factor. (I suspect Laplace transforms may be required for initial-value problems, while Fourier transforms are suitable for systems defined over all time, like asymptotic scattering states.)

At this point, it is useful to analyze the analytic structure of $\widetilde{K}(s)$ in Eq. (52). We can change variables $s \to -i\omega + \epsilon$, where ϵ is an infinitesimally small positive number, which we need if the integral in Eq. (49) is going to be guaranteed to converge. This yields:

$$
\begin{aligned}
\widetilde{K}(-i\omega + \epsilon) &= \int_{E_{th}}^{\infty} dE \, \frac{\rho(E)}{-i\omega + \epsilon + i(E - M)}, && (53) \\
&= i \int_{E_{th}}^{\infty} dE \, \frac{\rho(E)}{\omega - (E - M) + i\epsilon}, && (54) \\
&\equiv i\Sigma(\omega). && (55)
\end{aligned}
$$

We can then use the Sokhotski–Plemelj theorem in the limit that $\epsilon \to 0^+$:

$$
\Sigma(\omega) = \int_{E_{th}}^{\infty} dE \left[\mathcal{P} \frac{\rho(E)}{\omega - (E - M)} - i\pi\delta(\omega - (E - M))\rho(E) \right]. \quad (56)
$$

Here, $\mathcal{P}$ denotes the Cauchy principal value. This means:

$$
\begin{aligned}
\mathrm{Re}\Sigma(\omega) &= \int_{E_{th}}^{\infty} dE \, \mathcal{P} \frac{\rho(E)}{\omega - (E - M)} \equiv \Delta(\omega) && (57) \\
\mathrm{Im}\Sigma(\omega) &= -\pi\rho(\omega + M)\theta(\omega - (E_{th} - M)) \equiv -\frac{1}{2}\Gamma(\omega) && (58)
\end{aligned}
$$

This provides the interpretation that ω can be physically thought of as the energy difference between the parent particle and the final state continuum in the lab frame. We can note that, as defined, $\Gamma(\omega) \geq 0$, for all values of ω, since from Eq. (36), $\rho(E) \geq 0$ for all values of E. Also, $\Sigma(\omega)$ has a branch cut for $\omega > E_{th} - M$, where $\mathrm{Disc}\Sigma(\omega) = \Sigma(\omega + i\epsilon) - \Sigma(\omega - i\epsilon) = 2i\mathrm{Im}\Sigma(\omega) = -2\pi i\rho(\omega + M) = -i\Gamma(\omega)$ for $\omega > E_{th} - M$. It is possible that ρ may also contain contribution from bound states, but we'll ignore this contribution for now.

2nd order. Going back to Eq. (24) and considering the 2nd order correction:

$$
a^{(4)}(t) = (-i)^4 \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 \, \langle\psi| V(t_1)V(t_2)V(t_3)V(t_4) |\psi\rangle . \quad (59)
$$

Inserting the resolution of the identity, i.e., Eq. (27), in between $V(t_2)$ and $V(t_3)$, the only surviving term is:

$$
a^{(4)}(t) = (-i)^4 \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 \, \langle\psi| V(t_1)V(t_2) |\psi\rangle \langle\psi| V(t_3)V(t_4) |\psi\rangle . \quad (60)
$$

Using the definition of K in Eq. (40) and reordering the integration:

$$\begin{aligned}
a^{(4)}(t) &= (-i)^4 \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 \; K(t_1-t_2)K(t_3-t_4), && (61)\\
&= \int_0^t dt_1 \int_0^{t_1} d\tau_1 \int_0^{t_1-\tau_1} dt_3 \int_0^{t_3} d\tau_2 \; K(\tau_1)K(\tau_2), && (62)\\
&= \int_0^t dt_1 \int_0^{t_1} d\tau_1 \int_0^{t_1-\tau_1} d\tau_2 \; (t_1-\tau_1-\tau_2)K(\tau_1)K(\tau_2), && (63)\\
&= \int_0^t d\tau_1 \int_0^{t-\tau_1} d\tau_2 \int_{\tau_1+\tau 2}^{t} dt_1 \; (t_1-\tau_1-\tau_2)K(\tau_1)K(\tau_2), && (64)\\
&= \int_0^t d\tau_1 \int_0^{t-\tau_1} d\tau_2 \; \frac{(t-\tau_1-\tau_2)^2}{2} K(\tau_1)K(\tau_2). && (65)
\end{aligned}$$

Taking the Laplace transform:

$$\begin{aligned}
\widetilde{a}^{(4)}(s) &= \int_0^\infty dt\; e^{-st} a^{(4)}(t), && (66)\\
&= \frac{1}{2}\int_0^\infty dt \int_0^t d\tau_1 \int_0^{t-\tau_1} d\tau_2 \; e^{-st}(t-\tau_1-\tau_2)^2 K(\tau_1)K(\tau_2), && (67)\\
&= \frac{1}{2}\int_0^\infty d\tau_1 \int_0^\infty d\tau_2 \int_{\tau_1+\tau_2}^\infty dt\; e^{-st}(t-\tau_1-\tau_2)^2 K(\tau_1)K(\tau_2), && (68)\\
&= \frac{1}{s^3}\int_0^\infty d\tau_1 \int_0^\infty d\tau_2 \; e^{-s(\tau_1+\tau_2)} K(\tau_1)K(\tau_2), && (69)\\
&= \frac{1}{s^3}\widetilde{K}(s)^2. && (70)
\end{aligned}$$

Resummation. We will now resum the perturbative Dyson series to all orders. From the calculations of the Laplace transform of $a^{(2)}(t)$ and $a^{(4)}(t)$, we can infer the following pattern:

$$\widetilde{a}^{(2n)}(s) = \frac{(-1)^n}{s^{n+1}}\widetilde{K}(s)^n = \frac{1}{s}\left(-\frac{\widetilde{K}(s)}{s}\right)^n, \qquad (71)$$

so using the linearity of the Laplace transform, the Laplace transform of the Dyson series in Eq. (24) to all orders is:

$$\begin{aligned}
\widetilde{a}(s) &= \int_0^\infty dt\; e^{-st} a(t), && (72)\\
&= \sum_{n=0}^\infty \int_0^\infty dt\; e^{-st} a^{(2n)}(t), && (73)\\
&= \sum_{n=0}^\infty \widetilde{a}^{(2n)}(s), && (74)\\
&= \sum_{n=0}^\infty \frac{1}{s}\left(-\frac{\widetilde{K}(s)}{s}\right)^n, && (75)\\
&= \frac{1}{s+\widetilde{K}(s)}. && (76)
\end{aligned}$$

Taking the inverse Laplace transform:

$$a(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds\ e^{st}\ \widetilde{a}(s), \tag{77}$$

$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds\ \frac{e^{st}}{s+\widetilde{K}(s)}. \tag{78}$$

This is sometimes called the Mellin's inverse formula (and the contour is called the Bromwich contour). It is a vertical integration purely along the imaginary direction, for $\gamma > 0$, and γ has to be chosen so the integral converges. From Eq. (72), the Laplace transform converges as long as $\mathrm{Re}(s) > 0$. So, any $\gamma > 0$ will suffice for the inverse Laplace transform. In the following, we will pick the customary value, where $\gamma = \epsilon$, and ϵ is a infinitesimally small positive number. Changing variables $s = -i\omega + \epsilon$, the integral in Eq. (78) becomes:

$$a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\ \frac{e^{-i\omega t+\epsilon t}}{-i\omega+\epsilon+\widetilde{K}(-i\omega+\epsilon)}, \tag{79}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\ \frac{ie^{-i\omega t+\epsilon t}}{\omega+i\widetilde{K}(-i\omega+\epsilon)+i\epsilon}. \tag{80}$$

Using the definition for $i\Sigma(\omega) = \widetilde{K}(-i\omega+\epsilon)$ from Eq. (55), and dropping the innocuous $e^{\epsilon t}$ term:

$$a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\ e^{-i\omega t}\ \frac{i}{\omega-\Sigma(\omega)+i\epsilon}. \tag{81}$$

From Eq. (58), $\Sigma(\omega)$ has a branch cut for $\omega > E_{th} - M \equiv \omega_{th}$. Now we're going to deform the contour to wrap around the branch cut. To do so, first we will change variables $\omega \to \omega + i\epsilon$:

$$a(t) = \frac{1}{2\pi}\int_{-\infty+i\epsilon}^{\infty+i\epsilon} d\omega\ e^{-i\omega t}\ \frac{i}{\omega-\Sigma(\omega-i\epsilon)}. \tag{82}$$

From this change of variables, it's now clear that this integral traces out a path that is slightly above the branch cut. Now, we can add to it a contour at infinity in the lower half plane, where the integrand vanishes, yielding a closed contour. This contour then be deformed to wrap around the branch cut, so Eq. (82) becomes:

$$a(t) = \frac{1}{2\pi}\int_{\omega_{th}+i\epsilon}^{\infty+i\epsilon} d\omega\ e^{-i\omega t}\ \frac{i}{\omega-\Sigma(\omega-i\epsilon)} + \frac{1}{2\pi}\int_{\infty-i\epsilon}^{\omega_{th}-i\epsilon} d\omega\ e^{-i\omega t}\ \frac{i}{\omega-\Sigma(\omega+i\epsilon)}, \tag{83}$$

$$= \frac{1}{2\pi}\int_{\omega_{th}}^{\infty} d\omega\ e^{-i\omega t}\ i\mathrm{Disc}\left[\frac{1}{\omega-\Sigma(\omega)}\right], \tag{84}$$

$$= -\frac{1}{\pi}\int_{\omega_{th}}^{\infty} d\omega\ e^{-i\omega t}\ \mathrm{Im}\left[\frac{1}{\omega-\Sigma(\omega)}\right] \tag{85}$$

Using Eqs. (57) and (58), we can break up $\Sigma(\omega)$ into real and imaginary components $\Sigma(\omega) =$

$\Delta(\omega) - i\Gamma(\omega)/2$:

$$
\begin{aligned}
a(t) &= -\frac{1}{\pi}\int_{\omega_{th}}^{\infty} d\omega\ e^{-i\omega t}\ \text{Im}\left[\frac{1}{\omega - (\Delta(\omega) - i\Gamma(\omega)/2)}\right], && (86)\\
&= -\frac{1}{\pi}\int_{\omega_{th}}^{\infty} d\omega\ e^{-i\omega t}\ \text{Im}\left[\frac{\Delta(\omega) - i\Gamma(\omega)/2}{(\omega - \Delta(\omega))^2 + \Gamma(\omega)^2/4}\right], && (87)\\
&= \frac{1}{2\pi}\int_{\omega_{th}}^{\infty} d\omega\ e^{-i\omega t}\ \left[\frac{\Gamma(\omega)}{(\omega - \Delta(\omega))^2 + \Gamma(\omega)^2/4}\right]. && (88)
\end{aligned}
$$

Since $\Gamma(\omega)$ is non-negative, then the term in the square brackets is also non-negative. Since ω difference between the energy of the parent particle and the daughter state, we can change variables back to the energy $E = \omega + M$, where $E_{th} = \omega_{th} + M$ by definition:

$$
a(t) = \frac{e^{iMt}}{2\pi}\int_{E_{th}}^{\infty} dE\ e^{-iEt}\ \left[\frac{\Gamma(E)}{(E - M - \Delta(E))^2 + \Gamma(E)^2/4}\right], \tag{89}
$$

where we can do a simple shift redefinition of Δ and Γ from their definitions in Eqs. (57) and (58):

$$
\begin{aligned}
\Delta(E) &\equiv \int_{E_{th}}^{\infty} dE'\ \mathcal{P}\frac{\rho(E')}{E - E'}, && (90)\\
\Gamma(E) &= 2\pi\rho(E)\theta(E - E_{th}). && (91)
\end{aligned}
$$

Using Eq. (21) to translate back to the physical amplitude $A(t)$, the overall phase cancels, and we have:

$$
A(t) = \int_{E_{th}}^{\infty} dE\ e^{-iEt}\ f(E), \qquad f(E) \equiv \frac{1}{2\pi}\left[\frac{\Gamma(E)}{(E - M - \Delta(E))^2 + \Gamma(E)^2/4}\right]. \tag{92}
$$

where $f(E)$ is a non-negative function, i.e., $f(E) \geq 0$ for all real E. This is analogous to the equation T.6 in Sakurai [1].

2 The Breit-Wigner Approximation and Exponential Decay

The Breit-Wigner approximation makes the following assumption: $\Gamma(E) \ll M$, then the integral is dominated by the region $|E - M| \sim \Gamma$, and since this region is small, one can assume Δ and Γ are constants. In this case, the integral representation of $A(t)$ in Eq. (92) becomes:

$$
A(t) = \frac{1}{2\pi}\int_{E_{th}}^{\infty} dE\ e^{-iEt}\left[\frac{\Gamma}{(E - M - \Delta)^2 + \Gamma^2/4}\right] \tag{93}
$$

Defining the physical mass $M_{phys} = M + \Delta$:

$$
A(t) = \frac{1}{2\pi}\int_{E_{th}}^{\infty} dE\ e^{-iEt}\left[\frac{\Gamma}{(E - M_{phys})^2 + \Gamma^2/4}\right]. \tag{94}
$$

This is Eq. T.8a in Sakurai [1]. Now we can do the typical thing where we wave our hands and say something like: at the cost of negligible terms in limit $\Gamma \ll M$, and $\Gamma \ll E_{th}$, we can extend the lower limit of integration to $-\infty$ and perform the integral via contour integration:

$$\begin{aligned}
A(t) &\simeq \frac{1}{2\pi}\int_{-\infty}^{\infty} dE\; e^{-iEt}\left[\frac{\Gamma}{(E-M_{phys})^2+\Gamma^2/4}\right], &(95)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} dE\; e^{-iEt}\left[\frac{\Gamma}{(E-M_{phys}+i\Gamma/2)(E-M_{phys}-i\Gamma/2)}\right], &(96)\\
&= \frac{1}{2\pi}\oint_{LHP} dE\; e^{-iEt}\left[\frac{\Gamma}{(E-M_{phys}+i\Gamma/2)(E-M_{phys}-i\Gamma/2)}\right], &(97)\\
&= \left[\frac{1}{2\pi}\; e^{-iEt}\;\frac{-2\pi i\Gamma}{(E-M_{phys}-i\Gamma/2)}\right]\Bigg|_{E=M_{phys}-i\Gamma/2}, &(98)\\
&= e^{-iM_{phys}t-\Gamma t/2}. &(99)
\end{aligned}$$

So, the survival probability becomes:

$$P(t) = |A(t)|^2 = e^{-\Gamma t}. \tag{100}$$

This is pure exponential decay, and corresponds to Eq. T.8b in Sakurai [1].

3 Khalfin and the Paley-Wiener Theorem

In the above section on the Breit-Wigner approximation, two assumptions were made: (1) $\Gamma \ll M$ so the the support of the integrand is very narrow in energy, which justifies the approximations $\Gamma(E) \simeq \Gamma$ and $\Delta(E) \simeq \Delta$ as constants, and (2) one can extend the lower limit of the energy integral to $-\infty$ with negligible change to the final answer. These approximations are likely suitable for comparing to many experimental scenarios. However, extending the energy integral down to negative energies is nonphysical, and, as Sakurai mentions in Supplement II of Ref. [1], it violates analyticity. That is, purely exponential decay violates analyticity, and one should then expect non-exponential behavior of the survival probability at long times. However, no such deviation from exponential decay at long times has been observed experimentally.

This non-exponential tail of the survival probability was first pointed out by Khalfin in Ref. [3]. We can start with the original theorem by Paley and Wiener in their textbook [6], which is listed as Theorem XII on page 12:

> **Paley-Wiener Theorem**. Let $\phi(x)$ by a non-negative function not equivalent to zero, defined for $-\infty < x < \infty$, and of integral square in this range. A necessary and sufficient condition that there should exist a real- or complex-valued function $F(x)$ defined in the same range, vanishing for $x \geq x_0$ for some number x_0, and such that the Fourier transform $G(x)$ of $F(x)$ should satisfy $|G(x)| = \phi(x)$, is that
>
> $$\int_{-\infty}^{\infty} \frac{|\ln \phi(x)|}{1+x^2}\, dx < \infty. \tag{101}$$

To use this theorem, we have to show the following. (I will now set $E_{th} = 0$, since its precise value is moot for this discussion.)

1. While $A(t)$ is only defined for $t \geq 0$, one can uniquely define $A(t)$ for $-\infty < t < \infty$ via analytic continuation, so long as $f(E)$ does not grow exponentially as $E \to \infty$. To show this, one can establish the boundary condition:
$$A(t) = \lim_{\epsilon \to 0^+} A(t - i\epsilon), \tag{102}$$
where $\epsilon > 0$, and observe that
$$A(t - i\epsilon) = \int_0^\infty dE\ e^{-iEt}\ e^{-\epsilon E}\ f(E) \tag{103}$$
converges for any $\epsilon > 0$, so long as $f(E)$ does not grow exponentially as $E \to \infty$. This convergence holds even if $t < 0$. Therefore, we can consider $A(t)$ as the boundary value as $\epsilon \to 0^+$ of the function $A(t - i\epsilon)$, which is analytic in the lower-half plane. In the following, when referring to $A(t)$, it implies Eq. (102).

2. Let $\omega(E) = \theta(E)f(E)$. Here, $A(t)$ and $\omega(E)$ are Fourier transforms, i.e.,
$$A(t) = \int_0^\infty dE\ e^{-iEt}\ f(E) = \int_{-\infty}^\infty dE\ e^{-iEt}\ \omega(E), \tag{104}$$
$$\omega(E) = \frac{1}{2\pi}\int_{-\infty}^\infty dt\ e^{iEt}\ A(t). \tag{105}$$

3. $A(t)$ is square integrable for $-\infty < t < \infty$ if $f(E)$ is square integrable for $0 \leq E < \infty$. To show this, we can use the Plancherel theorem:
$$\int_{-\infty}^\infty dt\ |A(t)|^2 = 2\pi \int_{-\infty}^\infty dE\ |\omega(E)|^2 = 2\pi \int_0^\infty dE\ |f(E)|^2 \tag{106}$$
Therefore, if $f(E)$ is square integrable on the range $0 \leq E < \infty$, then $A(t)$ is square integrable for $-\infty < t < \infty$.

To proceed, we will assume that $f(E)$ is square integrable for $0 \leq E < \infty$ and $f(E)$ does not grow exponentially as $E \to \infty$. Now, we can use Payley-Wiener theorem, identifying G and F in the theorem to A and ω in our problem:
$$\int_{-\infty}^\infty \frac{|\ln|A(t)||}{1+t^2}\ dt < \infty. \tag{107}$$
This is the same bound claimed in Supplement II of Sakurai [1].

To see how this result makes purely exponential decay impossible, assume $A(t) = e^{-\Gamma t/2}$, where $\Gamma > 0$. Then Eq. (107) becomes:
$$\Gamma \int_0^\infty \frac{t}{1+t^2}\ dt < \infty. \tag{108}$$
This equation cannot be satisfied for any $\Gamma > 0$, since the integrand scales like $1/t$ as $t \to \infty$ and therefore the integral diverges. So, $A(t)$ cannot decay as an exponential for long times. The decay is "too fast", and some other functional contribution must dominate for asymptotically long times that decays more slowly than an exponential. This is the result first stated by Khalfin and reiterated in in Supplement II of Sakurai [1].

4 Asymptotic Expansion

Starting again with Eq. (92), we can do an asymptotic expansion to pick up subdominant behavior. The exact form of the survival amplitude in Eq. (92) can be expressed as the sum of two pieces:

$$A(t) = \underbrace{\int_{-\infty}^{\infty} dE\ e^{-iEt}\ f(E)}_{A_{BW}(t)} - \underbrace{\int_{-\infty}^{E_{th}} dE\ e^{-iEt}\ f(E)}_{R(t)} . \tag{109}$$

First, we'll look at the first term on the RHS, $A_{BW}(t)$. This is the standard Breit-Wigner term that leads to pure exponential decay, but we'll do the asymptotic expansion here just for completeness. Explicitly:

$$A_{BW}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dE\ e^{-iEt}\ \left[\frac{\Gamma(E)}{(E-M-\Delta(E))^2+\Gamma(E)^2/4}\right] . \tag{110}$$

This integral is dominated by the region around the value $E = E_0$, where E_0 satisfies $E_0 - M - \Delta(E_0) = 0$. So, we can let $x = E - E_0$, and we can expand about E_0, i.e.,

$$\Gamma(E) = \Gamma_0 + x\Gamma_0' + x^2\Gamma_0'' + \mathcal{O}(x^3), \tag{111}$$

$$\Delta(E) = \Delta_0 + x\Delta_0' + x^2\Delta_0'' + \mathcal{O}(x^3), \tag{112}$$

where $\Gamma_0 \equiv \Gamma(E_0)$, $\Gamma_0' \equiv \Gamma'(E_0)$, $\Delta_0 \equiv \Delta(E_0)$, etc. Switching integration variables from E to x, we then we have:

$$A_{BW}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\ e^{-i(E_0+x)t}\left[\frac{a_1+a_2x+a_3x^2}{c_1+c_2x+c_3x^2}\right] + \mathcal{O}(x^3), \tag{113}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\ e^{-i(E_0+x)t}\frac{1}{c_3}\left[\frac{a_1+a_2x+a_3x^2}{(x-\xi_+)(x-\xi_-)}\right], \tag{114}$$

where

$$a_1 = \Gamma_0 \tag{115}$$

$$a_2 = \Gamma_0' \tag{116}$$

$$a_3 = \Gamma_0'' \tag{117}$$

$$c_1 = \Gamma_0^2/4 \tag{118}$$

$$c_2 = \Gamma_0\Gamma_0'/2 \tag{119}$$

$$c_3 = (1-\Delta_0')^2 + \frac{1}{4}\left((\Gamma_0')^2 + \Gamma_0\Gamma_0''\right) \tag{120}$$

$$\xi_\pm = \frac{1}{2c_3}\left(-c_2 \pm \sqrt{c_2^2 - 4c_1c_3}\right) \tag{121}$$

We can note that the integral in Eq. (114) reduces to Eq. (96), where $\xi_\pm \to \pm i\Gamma_0/2$, in the limit that one ignores the E dependence of $\Delta(E)$ and $\Gamma(E)$, i.e., $\Gamma_0' = \Gamma_0'' = \Delta_0' = \Delta_0'' = 0$. In order to evaluate the integral in Eq. (114) using residue methods as before, we will assume that the magnitudes of Γ_0', Γ_0'', Δ_0', and Δ_0'' are small enough such that ξ_- sits below the real axis, and ξ_+ sits above the real axis. This way, when closing the contour in the lower-half plane, then we only pick up the pole due to ξ_-:

$$A_{BW}(t) = \frac{-2\pi i}{2\pi} e^{-i(E_0+\xi_-)t} \frac{1}{c_3} \left[\frac{a_1 + a_2\xi_- + a_3\xi_-^2}{(\xi_- - \xi_+)} \right], \tag{122}$$

$$= Be^{-\bar{\Gamma}t/2}, \tag{123}$$

where

$$B = -\frac{i}{c_3} \left[\frac{a_1 + a_2\xi_- + a_3\xi_-^2}{(\xi_- - \xi_+)} \right] e^{-i(E_0+\mathrm{Re}\xi_-)t}, \tag{124}$$

$$\bar{\Gamma} = -2\mathrm{Im}\xi_-. \tag{125}$$

The amplitude is still purely exponential decay, even with the subdominant terms included. Later on, imposing the normalization constraint that $|A(t)|^2 = 1$ as $t \to 0$ will fix the value of B.

Moving on to $R(t)$:

$$R(t) = \int_{-\infty}^{E_{th}} dE\ e^{-iEt} \left[\frac{\Gamma(E)}{(E - M - \Delta(E))^2 + \Gamma(E)^2/4} \right]. \tag{126}$$

We want to try to estimate this integral via asymptotic methods. This integral is dominated by the threshold region, i.e., around $E \sim E_{th}$. Note that Δ is regular around this region, however $\Gamma(E)$ is not, e.g, the square root behavior for 2-body decay from the phase space integral. That is:

$$\Delta(E) \sim \Delta(E_{th}), \quad E \to E_{th}, \tag{127}$$

$$\Gamma(E) \sim c(E - E_{th})^\alpha, \quad E \to E_{th}^+, \tag{128}$$

where c and α are constants. So, we will want to expand the integrand around the region $E \sim E_{th}$:

$$R(t) = \int_{-\infty}^{E_{th}} dE\ e^{-iEt} \left[\frac{c(E - E_{th})^\alpha}{(E_{th} - M - \Delta(E_{th}))^2 + c^2(E - E_{th})^{2\alpha}/4} \right]. \tag{129}$$

Change variables to $y = E_{th} - E$:

$$R(t) = e^{-iE_{th}t} \int_0^\infty dy\ e^{iyt} \left[\frac{c(-y)^\alpha}{(E_{th} - M - \Delta(E_{th}))^2 + c^2(-y)^{2\alpha}/4} \right], \tag{130}$$

$$\approx \frac{ce^{i\pi\alpha}e^{-iE_{th}t}}{D^2} \int_0^\infty dy\ e^{iyt}\ y^\alpha, \quad D \equiv E_{th} - M - \Delta(E_{th}) \tag{131}$$

$$= \frac{ce^{i\pi\alpha}e^{-iE_{th}t}}{D^2}\ e^{i\pi(\alpha+1)/2} \frac{\Gamma(\alpha+1)}{t^{\alpha+1}} + \mathcal{O}(t^{-(\alpha+2)}) \tag{132}$$

where now here $\Gamma(\alpha + 1)$ is the Γ-function. This term decays like a power of t, not like exponential decay.

Putting everything together, we have the survival probability:

$$P(t) = |A(t)|^2, \tag{133}$$

$$= |A_{BW}(t) - R(t)|^2, \tag{134}$$

$$= |A_{BW}(t)|^2 - 2\mathrm{Re}[A_{BW}^*(t)R(t)] + |R(t)|^2. \tag{135}$$

5 Numerics

Here we'll try some very back-of-the-envelope calculations to estimate the order of magnitude when the non-exponential decay may be observable. We'll make the following over-simplified assumptions, just to get a handle on the relative orders of magnitudes:

- Ignore the running of $\Gamma(E)$ and $\Delta(E)$. This means that $A_{BW}(t) = e^{-\Gamma_0 t/2}$, just like in Section 2. Here, we estimate $\Gamma_0 = \Gamma(M_{phys})$.
- Assume perfectly constructive interference between $A_{BW}(t)$ and $R(t)$. This means we can estimate that the interference term becomes as important as the pure exponential decay when $|A_{BW}(t)| \approx |R(t)|$. The value of t when this occurs is an estimate of the earliest possible time of an observable non-exponential decay. This is a best-case scenario, not a typical case.
- From the first bullet point, we can choose $M_{phys} = M + \Delta(E_0) \approx M + \Delta(E_{th})$.

With these assumptions, we want to solve for the value of $t_\star$ such that $|A_{BW}(t_\star)| = |R(t_\star)|$, where $A_{BW}(t)$ comes from Eq. (95) and $R(t)$ comes from Eq. (132):

$$e^{-\Gamma_0 t_\star/2} = \frac{c}{(E_{th} - M_{phys})^2} \frac{\Gamma(\alpha+1)}{t_\star^{\alpha+1}}. \tag{136}$$

As a reminder, the constants c and α come from the asymptotic expansion in Eq. (128). Here, Eq. (136) can be solved using special functions, but I don't want to attempt approximate solutions to this equation, since it is not clear a priori which factors are always large or small. Below are a few examples.

Muon decay. Here, $M_{phys} = m_\mu$ and $E_{th} = m_e$. The tree-level expression for $\Gamma(E)$ for muon decay is:

$$\Gamma(E) = \frac{G_F^2 E^5}{192\pi^3} \left(1 - 8x - 12x^2 \ln x + 8x^3 - x^4\right), \qquad x \equiv \frac{m_e^2}{E^2} \tag{137}$$

Using this function, we can make the following evaluations:

$$\Gamma_0 = \Gamma(m_\mu) \approx \frac{G_F^2 m_\mu^5}{192\pi^3} \tag{138}$$

$$\Gamma(E) \sim \frac{G_F^2}{15\pi^3}(E - m_e)^5, \qquad E \to m_e \tag{139}$$

so $\alpha = 5$ and $c = G_F^2/(15\pi^3)$. Here are the numbers we'll use:

$$\Gamma_0 = 3 \times 10^{-19} \text{ GeV} \tag{140}$$

$$c = 3 \times 10^{-13} \text{ GeV}^{-4} \tag{141}$$

$$E_{th} = 0.00051 \text{ GeV} \tag{142}$$

$$M_{phys} = 0.105 \text{ GeV} \tag{143}$$

$$\alpha = 5 \tag{144}$$

Plugging into Eq. (136), we have:

$$t_\star \approx 2.09 \times 10^{21}\ \text{GeV}^{-1} \tag{145}$$

or,

$$\Gamma_0 t_*/2 \approx 314 \text{ lifetimes} \tag{146}$$

This means one would have to wait about 300 lifetimes of the muon to see the non-exponential tail. This is likely beyond experimental reach.

Toy Model: $V \to \bar{\psi}\psi$. Here we'll look at a toy model for a vector decaying to two (colorless) fermions via the interaction $gV^\mu \bar{\psi}\gamma_\mu \psi$. Here, the vector boson has mass M and the fermions have mass m, so $M_{phys} = M$ and $E_{th} = 2m$. The tree-level expression for $\Gamma(E)$ for this decay is:

$$\Gamma(E) \;=\; \frac{g^2 E}{12\pi^2}\left(1 + \frac{2m^2}{E^2}\right)\sqrt{1 - \frac{4m^2}{E^2}} \tag{147}$$

where N_c is the number of colors. From this expression, we determine the following behaviors:

$$\Gamma_0 \;=\; \Gamma(M) = \frac{g^2 M}{12\pi^2}\left(1 + \frac{2m^2}{M^2}\right)\sqrt{1 - \frac{4m^2}{M^2}} \tag{148}$$

$$\Gamma(E) \;\sim\; \frac{g^2 m^{1/2}}{4\pi^2}(E - 2m)^{1/2}, \qquad E \to 2m \tag{149}$$

so $\alpha = 1/2$ and $c = g^2 m^{1/2}/(4\pi^2)$. Making up some numbers, let's choose $M = 0.25$ GeV, $m = 0.1$ GeV, and $g = 0.2$. So we have the following values:

$$\Gamma_0 \;=\; 6.7 \times 10^{-5}\ \text{GeV} \tag{150}$$

$$c \;=\; 3.2 \times 10^{-4}\ \text{GeV}^{1/2} \tag{151}$$

$$E_{th} \;=\; 0.2\ \text{GeV} \tag{152}$$

$$M_{phys} \;=\; 0.25\ \text{GeV} \tag{153}$$

$$\alpha \;=\; 1/2 \tag{154}$$

Plugging into Eq. (136), we have:

$$t_\star \approx 6.6 \times 10^{5}\ \text{GeV}^{-1} \tag{155}$$

or,

$$\Gamma_0 t_*/2 \approx 22 \text{ lifetimes} \tag{156}$$

This means one would have to wait about 20 lifetimes of V to see the non-exponential tail. The moral here is that if the mass of the parent particle decays to a state whose invariant mass is close in mass, and α is smaller, and Γ_0/M_{phys} is not very small, you can get into a regime where the non-exponential tail could be observable.

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